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A micromechanical damage model for effective elastoplastic behavior of partially debonded ductile matrix composites

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Abstract

A micromechanical damage model considering progressive partial debonding is presented to investigate the effective elastoplastic-damage behavior of partially debonded particle reinforced ductile matrix composites (PRDMCs). The effective, evolutionary elastoplastic-damage responses of three-phase composites, consisting of perfectly bonded spherical particles, partially debonded particles and a ductile matrix, are micromechanically derived on the basis of the ensemble-volume averaging procedure and the first-order effects of eigenstrains. The effects of random dispersion of particles are accommodated. Further, the evolutionary partial debonding mechanism is governed by the internal stresses of spherical particles and the statistical behavior of the interfacial strength. Specifically, following Zhao and Weng (1996), a partially debonded elastic spherical isotropic inclusion is replaced by an equivalent, transversely isotropic yet perfectly bonded elastic spherical inclusion. The Weibull's probabilistic function is employed to describe the varying probability of progressive partial particle debonding. The proposed effective yield criterion, together with the assumed overall associative plastic flow rule and the hardening law, forms the analytical framework for the estimation of the effective elastoplastic-damage behavior of ductile matrix composites. Finally, the present predictions are compared with the predictions based on Ju and Lee's (2000) complete particle debonding model, other existing numerical predictions, and available experimental data. It is observed that the effects of partially debonded particles on the stress–strain responses are significant when the damage evolution becomes rapid. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Micromechanics; Damage mechanics; Effective elastoplastic behavior; Partial interfacial debonding; Progressive damage; Metal matrix composites; Ensemble-volume averaging method; Random particle distribution

1. Introduction

Considerable work has been published in the literature on damage in particle-reinforced ductile matrix composites (PRDMCs). We refer to Dvorak (1991), Levy and Papazian (1991), Soboyejo et al. (1994), and

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Zhao and Weng (1995, 1996, 1997) for a literature review. Several failure mechanisms have been observed, such as the interfacial debonding between the matrix and inclusions (Lewis et al., 1993; Whitehouse and Clyne, 1993, 1995), the particle cracking (Lloyd, 1991), and the ductile plastic failure in the matrix (Lewandowski and Liu, 1989; Llorca et al., 1991). The mechanism of failure apparently depends on many factors, such as the interfacial strength, the strength of reinforcements, the manufacturing process, and the matrix aging condition (Lewandowski and Liu, 1989).

In recent years, Ju and Chen (1994a) and Ju and Tseng (1996, 1997) developed micromechanical formulations to predict effective elastoplastic behavior of two-phase metal matrix composites with random particle locations and under general loading histories. They considered the first-order and the second-order stress perturbations of elastic particles on the ductile matrix, and the second-order relationship between the far-field stress $\boldsymbol{\sigma}^0$ and the ensemble-volume averaged stress $\bar{\boldsymbol{\sigma}}$ on the basis of Ju and Chen (1994b, 1994c). On the other hand, Tohgo and Weng (1994) and Zhao and Weng (1995) proposed progressive interfacial complete debonding models for PRDMCs under triaxial tension. They used Weibull (1951) probability distribution function to describe the probability of complete interfacial particle debonding. It was postulated that the debonding of particles was controlled by the internal stresses of particles and the interfacial strength parameter. Further, Zhao and Weng (1996, 1997) derived effective elastic moduli and elastoplastic responses of partially debonded composites using fictitious, perfectly bonded transversely isotropic "equivalent" particles. Very recently, Ju and Lee (2000) proposed an elastoplastic-damage formulation based on a micromechanical framework and the ensemble-volume averaging approach for PRDMCs considering complete interfacial particle debonding. In particular, the authors predicted the overall elastoplastic behavior and damage evolution in three-phase PRDMCs based on the mechanical properties of constituent phases, particle volume fractions, random spatial inclusion distributions, micro-geometry of particles and probabilistic micromechanics.

The primary objective of the present paper is to extend the framework of Ju and Lee (2000) to assess the effects of partially debonded particle interfaces on the overall elastoplastic-damage behavior. The partial interfacial debonding mechanism in a PRDMC is displayed in Fig. 1. Specifically, when a PRDMC is subjected to a uniaxial tensile loading, the particles may partially debond on the top and bottom interfaces normal to the applied loading direction. The resulting partially debonded particles will lose their load-

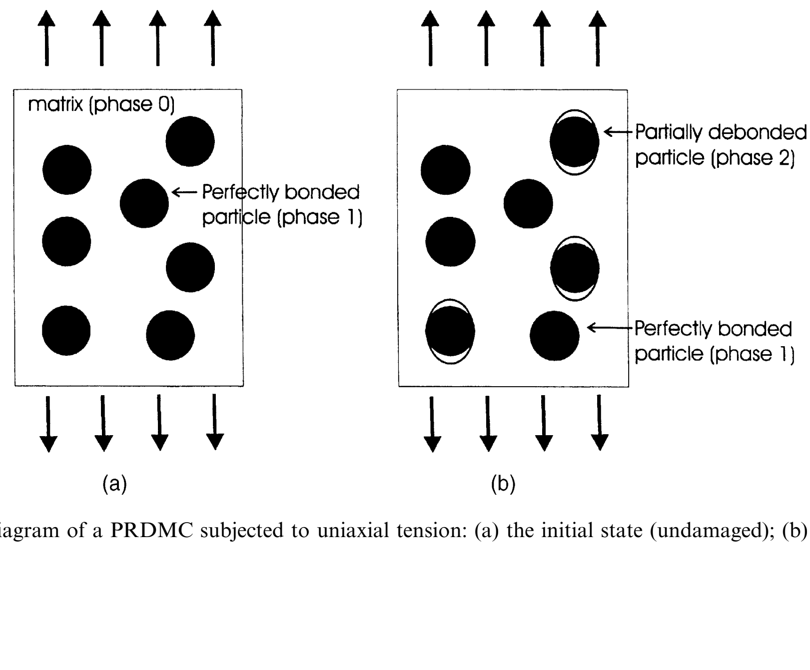

Fig. 1. A schematic diagram of a PRDMC subjected to uniaxial tension: (a) the initial state (undamaged); (b) the damaged state.

carrying capacity along the loading direction, but will still be able to transmit stresses to the matrix in the transverse direction through the bonded portion of the interfaces. In addition, the particles are assumed to be elastic spheres randomly dispersed in the matrix, and the ductile matrix behaves elastoplastically under uniaxial loading/unloading histories. All particles are assumed to be nonintersecting and initially embedded firmly in the matrix with perfect interfaces. It is further assumed that the partial interfacial debonding is governed by the average internal stress of a particle and the probabilistic Weibull's parameter of the particle–matrix interfacial strength.

This paper is organized as follows. In Section 2, we consider a transversely isotropic and perfectly bonded fictitious particle which is "equivalent" to a partially debonded isotropic particle. The effective elastic moduli of three-phase composites with perfectly bonded and partially debonded particles are micromechanically derived. In Section 3, the effective yield criterion and overall elastoplastic-damage characterization of three-phase composites are micromechanically constructed according to the ensemble-volume averaging procedure and the first-order eigenstrain effects owing to the randomly dispersed, perfectly bonded or partially debonded spherical particles. An evolutionary probabilistic interfacial partial particle debonding model is presented in Section 4 in accordance with the Weibull's function. The proposed probabilistic and progressive elastoplastic-damage formulation is applied to the uniaxial tensile loading in Section 5. Finally, to illustrate the potential applicability of the proposed method, the present predictions are compared with Ju and Lee's (2000) and Zhao and Weng's (1996) analytical predictions, and available experimental data in Section 6.

2. Effective elastic moduli of three-phase composites considering partial particle debonding

When a two-phase ductile matrix composite containing randomly dispersed, perfectly bonded spherical particles (see Fig. 1(a)) is subjected to remote uniaxial tensile loading, some particles may experience partial debonding on the "top" and "bottom" of the interfaces between the matrix and particles as deformations proceed (see Fig. 1(b)). A partially debonded particle will lose its load-carrying capacity along the debonded direction. Therefore, an initially two-phase composite would become a three-phase material, consisting of a ductile matrix, perfectly bonded particles, and partially debonded particles. In the initial state, the microstructure of a two-phase ductile matrix composite is assumed to be statistically homogeneous and isotropic, with a virtually constant volume fraction of particles. However, as deformations proceed under a uniaxial tension, the composite system progressively becomes transversely isotropic after evolutionary partial interfacial debonding. For simplicity, we shall assume that all partially debonded spherical particles are aligned.

Following Zhao and Weng (1996, 1997), a partially debonded isotropic spherical elastic particle is replaced by an equivalent, perfectly bonded spherical particle which possesses yet unknown transversely isotropic elastic moduli. The transverse isotropy of the "equivalent" (fictitious) particle can be determined in such a way that (a) its tensile and shear stresses will always vanish in the debonded direction, and (b) its stresses in the bonded directions exist as shown in Fig. 2 since the particle is still able to transmit stresses to the matrix along the bonded interfaces.

Let us start by considering a two-phase composite consisting of an elastoplastic matrix (phase 0) with elastic (three-dimensional) bulk modulus κ_0, elastic shear modulus μ_0, and randomly dispersed, perfectly bonded elastic spherical particles (phase 1) with (three-dimensional) bulk modulus κ_1 and shear modulus μ_0. As loading or deformations are applied, some particles could become partially debonded (phase 2), and the overall composite system is regarded as a transversely isotropic material. By designating the 1-direction as the axisymmetric axis and the plane 2–3 as the transversely isotropic plane, the stress–strain relation of a typical transversely isotropic solid can be written as

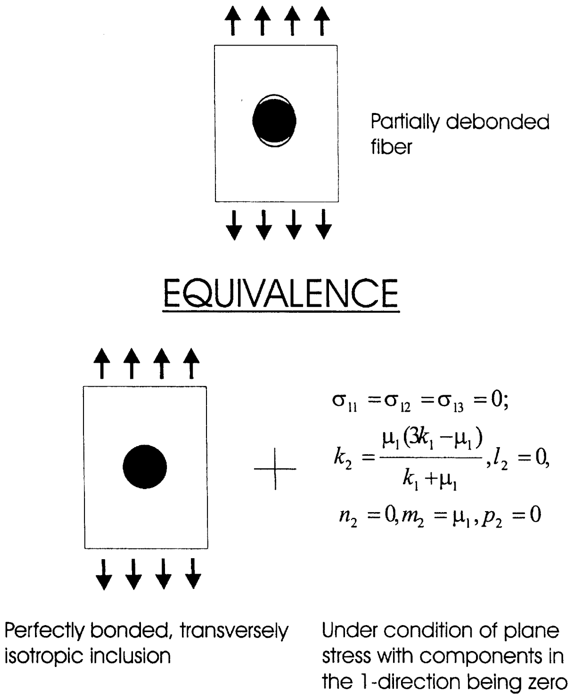

Fig. 2. A schematic representation of the equivalence between a partially debonded isotropic particle and an equivalent, perfectly bonded transversely isotropic particle.

$$
\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{pmatrix} \tag{1}
$$

The components of the stiffness matrix take the form:

$$
\frac{C_{22}+C_{23}}{2} = k, \quad C_{12} = l, \quad C_{11} = n, \quad \frac{C_{22}-C_{23}}{2} = C_{44} = m, \quad C_{55} = p \tag{2}
$$

where k is the plane stress bulk modulus for the lateral dilatation without longitudinal extension ($k = \kappa + \mu/3$); m is the rigidity modulus for shearing in any transverse direction; n denotes the modulus for the longitudinal uniaxial straining; l denotes the associated cross-modulus; and p signifies the axial shear modulus (Hill, 1964). Therefore, the stress–strain relationship for partially debonded composite can be rephrased as

$$
\begin{aligned}
&\frac{1}{2}(\sigma_{22}+\sigma_{33}) = k(\epsilon_{22}+\epsilon_{33}) + l\epsilon_{11} \\
&\sigma_{11} = l(\epsilon_{22}+\epsilon_{33}) + n\epsilon_{11} \\
&\sigma_{22}-\sigma_{33} = 2m(\epsilon_{22}-\epsilon_{33}) \\
&\sigma_{23} = 2m\epsilon_{23}, \quad \sigma_{12} = 2p\epsilon_{12}, \quad \sigma_{13} = 2p\epsilon_{13}
\end{aligned} \tag{3}
$$

It can be easily shown that, by using the inverse of the generalized Hook's law, the compliance matrix for a transversely isotropic material may be expressed as

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{pmatrix} = \begin{bmatrix} \frac{k}{l^2+kn} & \frac{l}{2(l^2-kn)} & \frac{l}{2(l^2-kn)} & 0 & 0 & 0 \\ \frac{l}{2(l^2-kn)} & \frac{-l^2+kn+mn}{4m(-l^2+kn)} & \frac{l^2-kn+mn}{4m(-l^2+kn)} & 0 & 0 & 0 \\ \frac{l}{2(l^2-kn)} & \frac{l^2-kn+mn}{4m(-l^2+kn)} & \frac{-l^2+kn+mn}{4m(-l^2+kn)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{m} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{p} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{p} \end{bmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} \tag{4}$$

For the special case of uniaxial loading σ_{11}, Eq. (4) can be simplified as

$$\epsilon_{ij} = \begin{bmatrix} \frac{k}{-l^2+kn} & 0 & 0 \\ 0 & \frac{l}{2(l^2-kn)} & 0 \\ 0 & 0 & \frac{l}{2(l^2-kn)} \end{bmatrix} \sigma_{11} \tag{5}$$

In Fig. 2, the transversely isotropic particle can be considered to be under the condition of plane stress with the components in the 1-direction being zero. To ensure the equivalence between a partially debonded isotropic particle and an equivalent, perfectly bonded transversely isotropic particle, the elastic moduli of a transversely isotropic particle, with the condition $\sigma_{11} = \sigma_{12} = \sigma_{13} = 0$, can be derived as

$$k_2 = \frac{\mu_1(3k_1 - \mu_1)}{k_1 + \mu_1}, \quad l_2 = 0, \quad n_2 = 0, \quad m_2 = \mu_1, \quad p_2 = 0 \tag{6}$$

where the subscripts 1 and 2 refer to the phases 1 and 2 moduli, respectively.

Effective elastic moduli of multi-phase composites containing randomly located, unidirectionally aligned ellipsoids were explicitly derived in Ju and Chen (1994b) accounting for far-field perturbations. For such a multi-phase composite, the (first-order) effective elastic stiffness tensor $\mathbf{C}_*$ reads

$$\mathbf{C}_* = \mathbf{C}_0 \cdot \left\{ \mathbf{I} + \mathbf{B} \cdot (\mathbf{I} - \mathbf{S} \cdot \mathbf{B})^{-1} \right\} \tag{7}$$

where $\mathbf{C}_0$ is the elasticity tensor of the matrix, $\mathbf{I}$ is the fourth-rank identity tensor, "·" denotes the tensor multiplication, and the fourth-rank tensor $\mathbf{B}$ is defined as

$$\mathbf{B} = \sum_{q=1}^{r} \phi_q (\mathbf{S} + \mathbf{A}_q)^{-1} \tag{8}$$

Here, r denotes the number of particle phases of different material properties and ϕ_q is the volume fraction of the q-phase. The components of the Eshelby's (1957) tensor $\mathbf{S}$ for a spherical inclusion embedded in an isotropic linear elastic and infinite matrix are

$$S_{ijkl} = \frac{1}{15(1-v_0)} \{(5v_0 - 1)\delta_{ij}\delta_{kl} + (4 - 5v_0)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\} \tag{9}$$

where v_0 is the Poisson's ratio of the matrix and δ_{ij} signifies the Kronecker delta. In addition, the fourth-rank tensor $\mathbf{A}_q$ is defined as

$$\mathbf{A}_q \equiv [\mathbf{C}_q - \mathbf{C}_0]^{-1} \cdot \mathbf{C}_0 \tag{10}$$

in which $\mathbf{C}_q$ is the elasticity tensor of the q-phase. It is noted that the matrix and particles in formula (7) could be isotropic or anisotropic, if the eigenstrain $\boldsymbol{\epsilon}^*(\mathbf{x})$ is uniform in a representative volume element

(RVE). In addition, the components of the Eshelby's tensor **S** depend on the Poisson's ratio of the matrix and the shape of the particle domain. Consequently, the components of the Eshelby's tensor **S** for a spherical transversely isotropic inclusion embedded in an isotropic linear elastic and infinite matrix are the same as those for a spherical isotropic inclusion given in Eq. (9).

For convenience, we introduce a transversely isotropic fourth-rank tensor $\tilde{\mathbf{F}}$ defined by six parameters b_m ($m = 1$–6):

$$\tilde{F}_{ijkl}(b_m) = b_1\tilde{n}_i\tilde{n}_j\tilde{n}_k\tilde{n}_l + b_2(\delta_{ik}\tilde{n}_j\tilde{n}_l + \delta_{il}\tilde{n}_j\tilde{n}_k + \delta_{jk}\tilde{n}_i\tilde{n}_l + \delta_{jl}\tilde{n}_i\tilde{n}_k) + b_3\delta_{ij}\tilde{n}_k\tilde{n}_l + b_4\delta_{kl}\tilde{n}_i\tilde{n}_j + b_5\delta_{ij}\delta_{kl} + b_6(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{11}$$

where $\tilde{\mathbf{n}}$ is the unit directional vector and the index m varies from 1 to 6. For the equivalent, transversely isotropic particle phase under consideration, the 1-direction is chosen as the axisymmetric direction, and therefore we have $\tilde{n}_1 = 1,\ \tilde{n}_2 = \tilde{n}_3 = 0$.

In accordance with the notation given in Eq. (2) and the above definition of $\tilde{\mathbf{F}}$, the stiffness tensor $\mathbf{C}_2$ for the equivalent, transversely isotropic particle can be represented as

$$\mathbf{C}_2 = \tilde{F}_{ijkl}(t_1, t_2, t_3, t_4, t_5, t_6) \tag{12}$$

where the six parameters on the right-hand side take the form:

$$t_1 = k_2 + n_2 + m_2 - 4p_2 - 2l_2 \tag{13}$$

$$t_2 = -m_2 + p_2 \tag{14}$$

$$t_3 = -k_2 + m_2 + l_2 \tag{15}$$

$$t_4 = -k_2 + m_2 + l_2 \tag{16}$$

$$t_5 = k_2 - m_2 \tag{17}$$

$$t_6 = m_2 \tag{18}$$

Since the inner product and inversion of a fourth-rank tensor $\tilde{\mathbf{F}}$ are the same as those of a fourth-rank tensor **F** given in Eq. (7) of Ju and Chen (1994c), we can derive the transversely isotropic fourth-rank tensor $(\mathbf{S} + \mathbf{A}_2)$ by using the formulas provided in the Appendix of Ju and Chen (1994c). The result is

$$\mathbf{S} + \mathbf{A}_2 = \frac{1}{30(1 - \nu_0)}\tilde{F}_{ijkl}(h_1, h_2, h_3, h_4, h_5, h_6) \tag{19}$$

where

$$h_1 = 60(1 - \nu_0)\frac{\mu_1}{\mu_0 - \mu_1}\frac{-3\kappa_0\kappa_1\mu_0 + 3\kappa_0\kappa_1\mu_1 - \kappa_0\mu_0\mu_1 + \kappa_1\mu_0\mu_1}{3\kappa_0\kappa_1\mu_0 - 3\kappa_0\kappa_1\mu_1 + 4\kappa_0\mu_0\mu_1 - 4\kappa_1\mu_0\mu_1} \tag{20}$$

$$h_2 = 15(1 - \nu_0)\frac{\mu_1}{\mu_0 - \mu_1} \tag{21}$$

$$h_3 = 60(1 - \nu_0)\frac{\mu_0\mu_1}{\mu_0 - \mu_1}\frac{\kappa_1\mu_1 - \kappa_0\mu_1}{3\kappa_0\kappa_1\mu_0 - 3\kappa_0\kappa_1\mu_1 + 4\kappa_0\mu_0\mu_1 - 4\kappa_0\mu_0\mu_1 - 4\kappa_1\mu_0\mu_1} \tag{22}$$

$$h_4 = 30(1 - \nu_0)\frac{\mu_1}{\mu_0 - \mu_1}\frac{3\kappa_0\kappa_1\mu_0 - 3\kappa_0\kappa_1\mu_1 - 2\kappa_0\mu_0\mu_1 + 2\kappa_1\mu_0\mu_1}{3\kappa_0\kappa_1\mu_0 - 3\kappa_0\kappa_1\mu_1 + 4\kappa_0\mu_0\mu_1 - 4\kappa_1\mu_0\mu_1} \tag{23}$$

$$h_5 = -60(1-\nu_0)\frac{\mu_0\mu_1}{\mu_0-\mu_1}\frac{\kappa_1\mu_0-\kappa_0\mu_1}{3\kappa_0\kappa_1\mu_0-3\kappa_0\kappa_1\mu_1+4\kappa_0\mu_0\mu_1-4\kappa_1\mu_0\mu_1}+2(5\nu_0-1) \tag{24}$$

$$h_6 = -15(1-\nu_0)\frac{\mu_0}{\mu_0-\mu_1}+2(4-5\nu_0) \tag{25}$$

The current three-phase composite consists of the matrix, perfectly bonded particles, and partially debonded particles (see Fig. 1). With the help of the tensor $\tilde{\mathbf{F}}$ defined in Eq. (11) together with the inner product and inversion of $\tilde{\mathbf{F}}$ given in the Appendix of Ju and Chen (1994c), components of the effective elastic stiffness tensor of the three-phase composite rendered by Eq. (7) can be explicitly derived as

$$\mathbf{C}_* = \tilde{F}_{ijkl}(c_1, c_2, c_3, c_4, c_5, c_6) \tag{26}$$

where the components $c_1, \ldots, c_6$ are listed in Appendix A. In addition, the five effective transverse elastic moduli read

$$k_* = c_5 + c_6, \quad l_* = c_4 + c_5, \quad m_* = c_6, \quad n_* = c_1 + 4c_2 + c_3 + c_4 + c_5 + 2c_6 + 2c_6, \quad p_* = c_2 + c_6 \tag{27}$$

3. Effective elastoplastic behavior of partially debonded composites

3.1. The stress norm and eigenstrain formulation

Let us now consider the effective elastoplastic responses of progressively and partially debonded particle composites. That is, an original two-phase composite may gradually become a three-phase composite consisting of the matrix, perfectly bonded particles and partially debonded particles. For simplicity, the von Mises yield criterion with isotropic hardening law is employed in the following. Extension of the present framework to general yield criterion and general hardening law is straightforward. At any matrix material point, the stress $\boldsymbol{\sigma}$ and the equivalent plastic strain $\bar{e}^{\mathrm{p}}$ must satisfy the following yield function:

$$F(\boldsymbol{\sigma}, \bar{e}^{\mathrm{p}}) = H(\boldsymbol{\sigma}) - K^2(\bar{e}^{\mathrm{p}}) \leqslant 0 \tag{28}$$

where $K(\bar{e}^{\mathrm{p}})$ is the isotropic hardening function of the matrix-only material, and $H(\boldsymbol{\sigma}) \equiv \boldsymbol{\sigma}:\mathbf{I}_{\mathrm{d}}:\boldsymbol{\sigma}$ designates the square of the deviatoric stress norm. Note that $\mathbf{I}_{\mathrm{d}}$ defines the deviatoric part of the fourth-rank identity tensor $\mathbf{I}$; i.e., $\mathbf{I}_{\mathrm{d}} \equiv \mathbf{I} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1}$, where $\mathbf{1}$ signifies the second-rank identity tensor and "$\otimes$" denotes the tensor expansion.

The total strain $\boldsymbol{\epsilon}$ can be decomposed into two parts; i.e., $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\mathrm{e}} + \boldsymbol{\epsilon}^{\mathrm{p}}$, where $\boldsymbol{\epsilon}^{\mathrm{e}}$ is the elastic strain of the matrix or particle, and $\boldsymbol{\epsilon}^{\mathrm{p}}$ defines the stress-free plastic strain in the plastic matrix only. In this study, an ensemble-volume averaged yield criterion is constructed for the three-phase composite. The methodology is parallel to Ju and Lee (2000) in which the first-order effects are considered in the effective plastic response. Moreover, small strains are assumed. Hence, the microstructure is taken as statistically homogeneous and isotropic with a virtually constant volume fraction for the summation of perfectly bonded and partially debonded particles during the deformation process. All particles are considered as spheres of uniform radius a.

Following Ju and Chen (1994a), Ju and Tseng (1996), and Ju and Lee (2000), $H(\mathbf{x}|\mathscr{G})$ denotes the square of the "current stress norm" at the local point $\mathbf{x}$, which determines the plastic strain in a PRDMC for a given phase configuration $\mathscr{G}$:

$$H(\mathbf{x}|\mathscr{G}) = \begin{cases} \boldsymbol{\sigma}(\mathbf{x}|\mathscr{G}) : \mathbf{I}_d : \boldsymbol{\sigma}(\mathbf{x}|\mathscr{G}) & \text{if } \mathbf{x} \text{ in the matrix} \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

Further, $\langle H\rangle_m(\mathbf{x})$ signifies the ensemble average of $H(\mathbf{x}|\mathscr{G})$ over all possible realizations where $\mathbf{x}$ is in the matrix phase. Let $P(\mathscr{G}_q)$ define the probability density function for finding the q-phase ($q = 1, 2$) configuration $\mathscr{G}_q$ in the composite. Accordingly, we have

$$\langle H\rangle_m(\mathbf{x}) = H^0 + \int_{\mathscr{G}_1} \left\{H(\mathbf{x}|\mathscr{G}_1) - H^0\right\}P(\mathscr{G}_1)\mathrm{d}\mathscr{G} + \int_{\mathscr{G}_2} \left\{H(\mathbf{x}|\mathscr{G}_2) - H^0\right\}P(\mathscr{G}_2)\mathrm{d}\mathscr{G} \tag{30}$$

where H^0 is the square of the far-field stress norm in the matrix; i.e., $H^0 = \boldsymbol{\sigma}^0{:}\mathbf{I}_\mathrm{d}{:}\boldsymbol{\sigma}^0$.

The total stress at any point $\mathbf{x}$ in the matrix is the superposition of the far-field stress $\boldsymbol{\sigma}^0$ and the perturbed stress $\boldsymbol{\sigma}'$ due to the presence of the particles; i.e, $\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}^0 + \boldsymbol{\sigma}'(\mathbf{x})$, where $\boldsymbol{\sigma}^0 \equiv \mathbf{C}_0{:}\boldsymbol{\epsilon}^0$ and $\boldsymbol{\sigma}'$ is defined as

$$\boldsymbol{\sigma}'(\mathbf{x}) \equiv \mathbf{C}_0 : \int_V \mathbf{G}(\mathbf{x}-\mathbf{x}') : \boldsymbol{\epsilon}_1^*(\mathbf{x}')\mathrm{d}\mathbf{x}' + \mathbf{C}_0 : \int_V \mathbf{G}(\mathbf{x}-\mathbf{x}') : \boldsymbol{\epsilon}_2^*(\mathbf{x}')\,\mathrm{d}\mathbf{x}' \tag{31}$$

Here, $\boldsymbol{\epsilon}^0$ corresponds to the elastic strain field induced by the far-field loading, $\boldsymbol{\epsilon}_q^*(\mathbf{x}')$ denotes the elastic eigenstrain in the q-phase ($q = 1, 2$), $\mathbf{x}'$ resides in a perfect or debonded particle, and V is the statistically representative volume element (infinitely large compared with inhomogeneities and without any prescribed displacement boundary conditions along infinite exterior boundaries). In indicial notation, the components of the fourth-rank Green's function tensor $\mathbf{G}$ are

$$G_{ijkl}(\mathbf{x}-\mathbf{x}') = \frac{1}{8\pi(1-v_0)r^3}\, F_{ijkl}(-15, 3v_0, 3, 3-6v_0, -1+2v_0, 1-2v_0) \tag{32}$$

where $\mathbf{r} \equiv \mathbf{x} - \mathbf{x}'$ and $r \equiv \|\mathbf{r}\|$. The components of the fourth-rank tensor $\mathbf{F}$ – which depends on six scalar quantities $B_1, B_2, B_3, B_4, B_5, B_6$ – are defined by

$$\begin{aligned} F_{ijkl}(B_m) \equiv{} & B_1 n_i n_j n_k n_l + B_2(\delta_{ik}n_j n_l + \delta_{il}n_j n_k + \delta_{jk}n_i n_l + \delta_{jl}n_i n_k) + B_3\delta_{ij}n_k n_l + B_4\delta_{kl}n_i n_j + B_5\delta_{ij}\delta_{kl} \\ & + B_6(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \end{aligned} \tag{33}$$

with the unit normal vector $\mathbf{n} \equiv \mathbf{r}/r$ and index $m = 1$–6.

The unknown elastic eigenstrain $\boldsymbol{\epsilon}_q^*(\mathbf{x})$ within the q-phase can be solved by the integral equation obtained from the celebrated Eshelby's equivalence principle (Eshelby, 1957). Similar to Ju and Lee (2000), the perturbed stress for any matrix point $\mathbf{x}$ due to a typical isolated q-phase inhomogeneity centered at $\mathbf{x}_q^{(1)}$ reads

$$\boldsymbol{\sigma}'(\mathbf{x}|\mathbf{x}_q^{(1)}) = [\mathbf{C}_0 \cdot \bar{\mathbf{G}}(\mathbf{x}-\mathbf{x}_q^{(1)})] : \boldsymbol{\epsilon}_q^{*0} \tag{34}$$

where $\boldsymbol{\epsilon}_q^{*0}$ is the solution of the (elastic) eigenstrain $\boldsymbol{\epsilon}_q^*$ for the single inclusion problem of the q-phase, and

$$\bar{\mathbf{G}}\left(\mathbf{x}-\mathbf{x}_q^{(1)}\right) \equiv \int_{\Omega_q^{(1)}} \mathbf{G}(\mathbf{x}-\mathbf{x}')\mathrm{d}\mathbf{x}' \tag{35}$$

for $\mathbf{x} \notin \Omega_q^{(1)}$ in which $\Omega_q^{(1)}$ is the single inhomogeneity domain centered at $\mathbf{x}_q^{(1)}$ in the q-phase. Moreover, the elastic "noninteracting" eigenstrain $\boldsymbol{\epsilon}_q^{*0}$ in Eq. (34) can be shown to be $\boldsymbol{\epsilon}_q^{*0} = -(\mathbf{A}_q + \mathbf{S})^{-1}{:}\boldsymbol{\epsilon}^0$ with $q = 1$,2; cf. Ju and Chen (1994b,c).

3.2. Effective elastoplastic characterization of partially debonded particulate composites

Since a matrix point receives the perturbations from perfectly bonded particles and from partially debonded particles, the ensemble-average stress norm for any matrix point $\mathbf{x}$ can be evaluated by collecting and summing up all perturbations produced by any typical perfectly bonded particle centered at $\mathbf{x}_1^{(1)}$ and any partially debonded particle centered at $\mathbf{x}_2^{(1)}$, and averaging over all possible locations of $\mathbf{x}_1^{(1)}$ and $\mathbf{x}_2^{(1)}$. As a result, we obtain

$$\langle H\rangle_m(\mathbf{x}) \cong H^0 + \int_{|\mathbf{x}-\mathbf{x}_1^{(1)}|>a} \left\{H\left(\mathbf{x}|\mathbf{x}_1^{(1)}\right) - H^0\right\} P\left(\mathbf{x}_1^{(1)}\right) \mathrm{d}\mathbf{x}^{(1)} + \int_{|\mathbf{x}-\mathbf{x}_2^{(1)}|>a} \left\{H\left(\mathbf{x}|\mathbf{x}_2^{(1)}\right) - H^0\right\} P\left(\mathbf{x}_2^{(1)}\right) \mathrm{d}\mathbf{x}^{(1)} + \cdots \tag{36}$$

where $P(\mathbf{x}_1^{(1)})$ and $P(\mathbf{x}_2^{(1)})$ signify the probability density function for finding a perfectly bonded particle centered at $\mathbf{x}_1^{(1)}$ and a partially debonded particle centered at $\mathbf{x}_2^{(1)}$, respectively. For brevity, $P(\mathbf{x}_1^{(1)})$ and $P(\mathbf{x}_2^{(1)})$ are assumed to be statistically homogeneous, isotropic and uniform. That is, we have $P(\mathbf{x}_1^{(1)}) = N_1/V$ and $P(\mathbf{x}_2^{(1)}) = N_2/V$, where N_1 and N_2 are the total number of perfectly bonded particles and partially debonded particles, respectively, dispersed in a representative volume V. Moreover, due to the assumption of statistical isotropy and uniformity, Eq. (36) can be recast into a more convenient form:

$$\langle H\rangle_m(\mathbf{x}) \cong H^0 + \frac{N_1}{V}\int_{\hat{r}_1>a} \mathrm{d}\hat{r}_1 \int_{A(\hat{r}_1)} \left\{H(\hat{\mathbf{r}}_1) - H^0\right\} \mathrm{d}A + \frac{N_2}{V}\int_{\hat{r}_2>a} \mathrm{d}\hat{r}_2 \int_{A(\hat{r}_2)} \left\{H(\hat{\mathbf{r}}_2) - H^0\right\} \mathrm{d}A + \cdots \tag{37}$$

Here, $A(\hat{r}_q)$ denotes a spherical surface of radius $\hat{r}_q$ $(q = 1, 2)$, with $\hat{\mathbf{r}}_q = \mathbf{x} - \mathbf{x}_q^{(1)}$ and $\hat{r}_q \equiv \|\hat{\mathbf{r}}_q\|$.

In addition, to obtain the perturbed stress $\boldsymbol{\sigma}'(\mathbf{x}|\mathbf{x}_q^{(1)})$, we can perform the inner product of $\hat{\mathbf{F}}(u_m)$ defined in Eq. (29) of Ju and Lee (2000) and $\tilde{\mathbf{F}}(v_m)$ previously defined in Eq. (11). Specifically, the components of the fourth-rank tensor F^* – which depends on seventeen scalar quantities $w_1, \ldots, w_{17}$ – are defined as

$$\begin{aligned}
F^*_{ijkl}(w_m) &\equiv \hat{F}_{ijpq}(u_m)\cdot\tilde{F}_{pqkl}(v_m) \\
&= [u_1\hat{n}_i\hat{n}_j\hat{n}_p\hat{n}_q + u_2(\delta_{ip}\hat{n}_j\hat{n}_q + \delta_{iq}\hat{n}_j\hat{n}_p + \delta_{jp}\hat{n}_i\hat{n}_q + \delta_{jq}\hat{n}_i\hat{n}_p) + u_3\delta_{ij}\hat{n}_p\hat{n}_q \\
&\qquad + u_4\delta_{pq}\hat{n}_i\hat{n}_j + u_5\delta_{ij}\delta_{pq} + u_6(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp})]\cdot \\
&\quad [v_1\tilde{n}_p\tilde{n}_q\tilde{n}_k\tilde{n}_l + v_2(\delta_{pk}\tilde{n}_q\tilde{n}_l + \delta_{pl}\tilde{n}_q\tilde{n}_k + \delta_{qk}\tilde{n}_p\tilde{n}_l + \delta_{ql}\tilde{n}_p\tilde{n}_k) + v_3\delta_{pq}\tilde{n}_k\tilde{n}_l \\
&\qquad + v_4\delta_{kl}\tilde{n}_p\tilde{n}_q + v_5\delta_{pq}\delta_{kl} + v_6(\delta_{pk}\delta_{ql} + \delta_{pl}\delta_{qk})] \\
&= w_1\delta_{kl}\hat{n}_i\hat{n}_j + w_2\delta_{ij}\hat{n}_k\hat{n}_l + w_3\tilde{n}_k\tilde{n}_l\hat{n}_i\hat{n}_j + w_4\hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_l + w_5\delta_{ij}\tilde{n}_k\tilde{n}_l + w_6\delta_{kl}\tilde{n}_i\tilde{n}_j + w_7\delta_{ij}\delta_{kl} \\
&\quad + w_8\tilde{n}_i\tilde{n}_j\tilde{n}_k\tilde{n}_l + w_9(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + w_{10}(\delta_{ik}\tilde{n}_j\tilde{n}_l + \delta_{il}\tilde{n}_j\tilde{n}_k + \delta_{jk}\tilde{n}_i\tilde{n}_l + \delta_{jl}\tilde{n}_i\tilde{n}_k) \\
&\quad + w_{11}(\delta_{ij}\hat{n}_k\tilde{n}_l + \delta_{ij}\hat{n}_l\tilde{n}_k) + w_{12}(\tilde{n}_i\tilde{n}_k\hat{n}_j\hat{n}_l + \tilde{n}_i\tilde{n}_l\hat{n}_j\hat{n}_k + \tilde{n}_j\tilde{n}_k\hat{n}_i\hat{n}_l + \tilde{n}_j\tilde{n}_l\hat{n}_i\hat{n}_k) \\
&\quad + w_{13}(\delta_{ik}\hat{n}_j\hat{n}_l + \delta_{il}\hat{n}_j\hat{n}_k + \delta_{jk}\hat{n}_i\hat{n}_l + \delta_{jl}\hat{n}_i\hat{n}_k) + w_{14}(\delta_{ik}\hat{n}_j\tilde{n}_l + \delta_{il}\hat{n}_j\tilde{n}_k + \delta_{jk}\hat{n}_i\tilde{n}_l + \delta_{jl}\hat{n}_i\tilde{n}_k) \\
&\quad + w_{15}(\delta_{kl}\hat{n}_i\tilde{n}_j + \delta_{kl}\hat{n}_j\tilde{n}_i) + w_{16}(\tilde{n}_k\tilde{n}_l\hat{n}_i\tilde{n}_j + \tilde{n}_k\tilde{n}_l\hat{n}_j\tilde{n}_i) + w_{17}(\hat{n}_i\hat{n}_j\hat{n}_l\tilde{n}_k + \hat{n}_i\hat{n}_j\hat{n}_k\tilde{n}_l)
\end{aligned} \tag{38}$$

where $\hat{\mathbf{n}} \equiv \hat{\mathbf{r}}_q/\hat{r}_q$ denotes the unit outer normal vector, $\tilde{\mathbf{n}}$ is the unit directional vector, and the indices $m = 1$–6, $n = 1$–17. The parameters $w_1, \ldots, w_{17}$ are given in Appendix B.

During the ensemble average evaluation of the surface integrals, the following five different identity groups are discovered.

(i) When the integrand $[H(\hat{\mathbf{r}}_2) - H^0]$ does not have $\hat{n}_i$, we arrive at

$$\int_{A(\hat{r}_2)} \mathrm{d}A = 4\pi\hat{r}_2^2 \tag{39}$$

$$\int_{A(\hat{r}_2)} \hat{n}_1^2\,\mathrm{d}A = \frac{4\pi\hat{r}_2^2}{3} \tag{40}$$

$$\int_{A(\hat{r}_2)} \hat{n}_1^4\,\mathrm{d}A = \frac{4\pi\hat{r}_2^2}{5} \tag{41}$$

$$\int_{A(\hat{r}_2)} \hat{n}_1^m \,\mathrm{d}A = 0, \quad m = 1, 3, 5 \tag{42}$$

(ii) When the integrand $[H(\hat{\mathbf{r}}_2) - H^0]$ has $\hat{n}_i$, we have

$$\int_{A(\hat{r}_2)} \hat{n}_1 \hat{n}_i \,\mathrm{d}A = \frac{4\pi\hat{r}_2^2}{3} \tilde{n}_i \tag{43}$$

$$\int_{A(\hat{r}_2)} \hat{n}_1^3 \hat{n}_i \,\mathrm{d}A = \frac{4\pi\hat{r}_2^2}{5} \tilde{n}_i \tag{44}$$

$$\int_{A(\hat{r}_2)} \hat{n}_1^m \hat{n}_i \,\mathrm{d}A = 0, \quad m = 0, 2, 4 \tag{45}$$

(iii) When the integrand $[H(\hat{\mathbf{r}}_2) - H^0]$ has $\hat{n}_i\hat{n}_j$, we obtain

$$\int_{A(\hat{r}_2)} \hat{n}_i \hat{n}_j \,\mathrm{d}A = \frac{4\pi\hat{r}_2^2}{3} \delta_{ij} \tag{46}$$

$$\int_{A(\hat{r}_2)} \hat{n}_1^2 \hat{n}_i \hat{n}_j \,\mathrm{d}A = \frac{4\pi\hat{r}_2^2}{15} (2\tilde{n}_i\tilde{n}_j + \delta_{ij}) \tag{47}$$

$$\int_{A(\hat{r}_2)} \hat{n}_1^4 \hat{n}_i \hat{n}_j \,\mathrm{d}A = \frac{4\pi\hat{r}_2^2}{35} (4\tilde{n}_i\tilde{n}_j + \delta_{ij}) \tag{48}$$

$$\int_{A(\hat{r}_2)} \hat{n}_1^m \hat{n}_i \hat{n}_j \,\mathrm{d}A = 0, \quad m = 1, 3 \tag{49}$$

(iv) When the integrand $[H(\hat{\mathbf{r}}_2) - H^0]$ has $\hat{n}_i\hat{n}_j\hat{n}_k$, we write

$$\int_{A(\hat{r}_2)} \hat{n}_1 \hat{n}_i \hat{n}_j \hat{n}_k \,\mathrm{d}A = \frac{4\pi\hat{r}_2^2}{15} (2\tilde{n}_i\tilde{n}_j + \delta_{ij})\tilde{n}_k \tag{50}$$

$$\int_{A(\hat{r}_2)} \hat{n}_1^3 \hat{n}_i \hat{n}_j \hat{n}_k \,\mathrm{d}A = \frac{4\pi\hat{r}_2^2}{35} (4\tilde{n}_i\tilde{n}_j + \delta_{ij})\tilde{n}_k \tag{51}$$

$$\int_{A(\hat{r}_2)} \hat{n}_1^m \hat{n}_i \hat{n}_j \hat{n}_k \,\mathrm{d}A = 0, \quad m = 0, 2, 4 \tag{52}$$

(v) When the integrand $[H(\hat{\mathbf{r}}_2) - H^0]$ has $\hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_l$, we obtain

$$\int_{A(\hat{r}_2)} \hat{n}_i \hat{n}_j \hat{n}_k \hat{n}_l \,\mathrm{d}A = \frac{4\pi\hat{r}_2^2}{15} (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{53}$$

By using the above five identity groups, the perturbed stress rendered by Eq. (34) and the inner product of the fourth-rank tensor $\mathbf{F}^*$ in Appendix C, the ensemble-averaged current stress norm at any matrix point (see Eq. (37)) is derived as

$$\langle H \rangle_m(\mathbf{x}) = \boldsymbol{\sigma}^0 : \mathbf{T}^\mathrm{p} : \boldsymbol{\sigma}^0 \tag{54}$$

Here, the components of the positive definite fourth-rank tensor $\mathbf{T}^\mathrm{p}$ read

$$T^{\rm p}_{ijkl} = T^{\rm p}_1\tilde{n}_i\tilde{n}_j\tilde{n}_k\tilde{n}_l + T^{\rm p}_2(\delta_{ik}\tilde{n}_j\tilde{n}_l + \delta_{il}\tilde{n}_j\tilde{n}_k + \delta_{jk}\tilde{n}_i\tilde{n}_l + \delta_{jl}\tilde{n}_i\tilde{n}_k) + T^{\rm p}_3\delta_{ij}\tilde{n}_k\tilde{n}_l + T^{\rm p}_4\delta_{kl}\tilde{n}_i\tilde{n}_j + T^{\rm p}_5\delta_{ij}\delta_{kl} + T^{\rm p}_6(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{55}$$

in which the parameters $T^{\rm p}_1, \ldots, T^{\rm p}_6$ are rendered in Appendix D.

The ensemble-averaged current stress norm at a matrix point can be expressed in terms of the macroscopic stress $\bar{\boldsymbol{\sigma}}$. Similar to Ju and Chen (1994a), the relation between the far-field stress $\boldsymbol{\sigma}^0$ and the macroscopic stress $\bar{\boldsymbol{\sigma}}$ takes the form

$$\boldsymbol{\sigma}^0 = \mathbf{P}^{\rm p} : \bar{\boldsymbol{\sigma}} \tag{56}$$

where the components of $\mathbf{P}^{\rm p}$ read

$$P^{\rm p}_{ijkl} = P^{\rm p}_1\tilde{n}_i\tilde{n}_j\tilde{n}_k\tilde{n}_l + P^{\rm p}_2(\delta_{ik}\tilde{n}_j\tilde{n}_l + \delta_{il}\tilde{n}_j\tilde{n}_k + \delta_{jk}\tilde{n}_i\tilde{n}_l + \delta_{jl}\tilde{n}_i\tilde{n}_k) + P^{\rm p}_3\delta_{ij}\tilde{n}_k\tilde{n}_l + P^{\rm p}_4\delta_{kl}\tilde{n}_i\tilde{n}_j + P^{\rm p}_5\delta_{ij}\delta_{kl} + P^{\rm p}_6(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{57}$$

and the parameters $P^{\rm p}_1, \ldots, P^{\rm p}_6$ are summarized in Appendix E.

Combination of Eqs. (54) and (56) then leads to the following expression for the ensemble-averaged current stress norm (square) in a matrix point:

$$\langle H \rangle_m(\mathbf{x}) = \bar{\boldsymbol{\sigma}} : \bar{\mathbf{T}}^{\rm p} : \bar{\boldsymbol{\sigma}} \tag{58}$$

where the positive definite fourth-rank tensor $\bar{\mathbf{T}}^{\rm p}$ is defined as

$$\bar{\mathbf{T}}^{\rm p} \equiv (\mathbf{P}^{\rm p})^{\rm T} \cdot \mathbf{T}^{\rm p} \cdot \mathbf{P}^{\rm p} \tag{59}$$

and takes the form

$$\bar{T}^{\rm p}_{ijkl} = \bar{T}^{\rm p}_1\tilde{n}_i\tilde{n}_j\tilde{n}_k\tilde{n}_l + \bar{T}^{\rm p}_2(\delta_{ik}\tilde{n}_j\tilde{n}_l + \delta_{il}\tilde{n}_j\tilde{n}_k + \delta_{jk}\tilde{n}_i\tilde{n}_l + \delta_{jl}\tilde{n}_i\tilde{n}_k) + \bar{T}^{\rm p}_3\delta_{ij}\tilde{n}_k\tilde{n}_l + \bar{T}^{\rm p}_4\delta_{kl}\tilde{n}_i\tilde{n}_j + \bar{T}^{\rm p}_5\delta_{ij}\delta_{kl} + \bar{T}^{\rm p}_6(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{60}$$

where

$$\begin{aligned}\bar{T}^{\rm p}_1 = {} & 4P^{\rm p}_1P^{\rm p}_6T^{\rm p}_6 + 8(P^{\rm p}_1 + 2P^{\rm p}_2 + P^{\rm p}_3)(P^{\rm p}_2T^{\rm p}_2 + P^{\rm p}_6T^{\rm p}_2 + P^{\rm p}_2T^{\rm p}_6) + (P^{\rm p}_1 + 4P^{\rm p}_2 + 3P^{\rm p}_3) \\ & \times (P^{\rm p}_1T^{\rm p}_4 + 4P^{\rm p}_2T_4 + P^{\rm p}_4T^{\rm p}_4 + 2P^{\rm p}_6T^{\rm p}_4 + P^{\rm p}_1T^{\rm p}_5 + 4P^{\rm p}_2T^{\rm p}_5 + 3P^{\rm p}_4T^{\rm p}_5 + 2P^{\rm p}_4T^{\rm p}_6) + (P^{\rm p}_1 + 4P^{\rm p}_2 + P^{\rm p}_3 + 2P^{\rm p}_6) \\ & \times [2P^{\rm p}_6T^{\rm p}_1 + 4P^{\rm p}_2(T^{\rm p}_1 + 2T^{\rm p}_2 + T^{\rm p}_3) + P^{\rm p}_4(T^{\rm p}_1 + 4T^{\rm p}_2 + 3T^{\rm p}_3) + P^{\rm p}_1(T^{\rm p}_1 + 4T^{\rm p}_2 + T^{\rm p}_3 + 2T^{\rm p}_6)]\end{aligned} \tag{61}$$

$$\bar{T}^{\rm p}_2 = 4(P^{\rm p}_2P^{\rm p}_2T^{\rm p}_2 + 2P^{\rm p}_2P^{\rm p}_6T^{\rm p}_2 + P^{\rm p}_6P^{\rm p}_6T^{\rm p}_2 + P^{\rm p}_2P^{\rm p}_2T^{\rm p}_6 + 2P^{\rm p}_2P^{\rm p}_6T^{\rm p}_6) \tag{62}$$

$$\begin{aligned}\bar{T}^{\rm p}_3 = {} & 4P^{\rm p}_3P^{\rm p}_6T^{\rm p}_6 + (P^{\rm p}_1 + 4P^{\rm p}_2 + 3P^{\rm p}_3)(P^{\rm p}_3T^{\rm p}_4 + P^{\rm p}_5T^{\rm p}_4 + P^{\rm p}_3T^{\rm p}_5 + 3P^{\rm p}_5T^{\rm p}_5 + 2P^{\rm p}_6T^{\rm p}_5 + 2P^{\rm p}_5T^{\rm p}_6) \\ & + (P^{\rm p}_1 + 4P^{\rm p}_2 + P^{\rm p}_3 + 2P^{\rm p}_6)[2P^{\rm p}_6T^{\rm p}_3 + P^{\rm p}_5(T^{\rm p}_1 + 4T^{\rm p}_2 + 3T^{\rm p}_3) + P^{\rm p}_3(T^{\rm p}_1 + 4T^{\rm p}_2 + T^{\rm p}_3 + 2T^{\rm p}_6)]\end{aligned} \tag{63}$$

$$\begin{aligned}\bar{T}^{\rm p}_4 = {} & 4P^{\rm p}_4P^{\rm p}_6T^{\rm p}_6 + 8(P^{\rm p}_4 + P^{\rm p}_5)(P^{\rm p}_2T^{\rm p}_2 + P^{\rm p}_6T^{\rm p}_2 + P^{\rm p}_2T^{\rm p}_6) + (P^{\rm p}_4 + 3P^{\rm p}_5 + 2P^{\rm p}_6)(P^{\rm p}_1T^{\rm p}_4 + 4P^{\rm p}_2T^{\rm p}_4 + P^{\rm p}_4T^{\rm p}_4 \\ & + 2P^{\rm p}_6T^{\rm p}_4 + P^{\rm p}_1T^{\rm p}_5 + 4P^{\rm p}_2T^{\rm p}_5 + 3P^{\rm p}_4T^{\rm p}_5 + 2P^{\rm p}_4T^{\rm p}_6) + (P^{\rm p}_4 + P^{\rm p}_5)[2P^{\rm p}_6T^{\rm p}_1 + 4P^{\rm p}_2(T^{\rm p}_1 + 2T^{\rm p}_2 + T^{\rm p}_3) \\ & + P^{\rm p}_4(T^{\rm p}_1 + 4T^{\rm p}_2 + 3T^{\rm p}_3) + P^{\rm p}_1(T^{\rm p}_1 + 4T^{\rm p}_2 + T^{\rm p}_3 + 2T^{\rm p}_6)]\end{aligned} \tag{64}$$

$$\begin{aligned}\bar{T}^{\rm p}_5 = {} & 4P^{\rm p}_5P^{\rm p}_6T^{\rm p}_6 + (P^{\rm p}_4 + 3P^{\rm p}_5 + 2P^{\rm p}_6)(P^{\rm p}_3T^{\rm p}_4 + P^{\rm p}_5T^{\rm p}_4 + P^{\rm p}_3T^{\rm p}_5 + 3P^{\rm p}_5T^{\rm p}_5 + 2P^{\rm p}_6T^{\rm p}_5 + 2P^{\rm p}_5T^{\rm p}_6) \\ & + (P^{\rm p}_4 + P^{\rm p}_5)[2P^{\rm p}_6T^{\rm p}_3 + P^{\rm p}_5(T^{\rm p}_1 + 4T^{\rm p}_2 + 3T^{\rm p}_3) + P^{\rm p}_3(T^{\rm p}_1 + 4T^{\rm p}_2 + T^{\rm p}_3 + 2T^{\rm p}_6)]\end{aligned} \tag{65}$$

$$\bar{T}^{\rm p}_6 = 4P^{\rm p}_6P^{\rm p}_6T^{\rm p}_6 \tag{66}$$

Furthermore, the ensemble-volume averaged "current stress norm" for any point $\mathbf{x}$ in a three-phase particulate composite can be defined as

$$\sqrt{\langle H\rangle(\mathbf{x})} = (1-\phi_1)\sqrt{\bar{\boldsymbol{\sigma}} : \bar{\mathbf{T}}^{\rm p} : \bar{\boldsymbol{\sigma}}} \tag{67}$$

where ϕ_1 denotes the current volume fraction of perfectly bonded particles. Therefore, the effective yield function for the three-phase PRDMC can be proposed as

$$\bar{F} = (1-\phi_1)^2 \bar{\boldsymbol{\sigma}} : \bar{\mathbf{T}}^{\rm p} : \bar{\boldsymbol{\sigma}} - K^2(\bar{e}^{\rm p}) \tag{68}$$

with the isotropic hardening function $K(\bar{e}^{\rm p})$ for the composite. It is noted that the effective yield function is pressure dependent and not of the von Mises type. Moreover, for simplicity, we assume that the overall flow rule for the matrix is associative. Accordingly, the effective ensemble-volume averaged plastic strain rate for the PRDMC can be expressed as

$$\dot{\bar{\boldsymbol{\epsilon}}}^{\rm p} = \dot{\lambda}\frac{\partial \bar{F}}{\partial \bar{\boldsymbol{\sigma}}} = 2(1-\phi_1)^2 \dot{\lambda} \bar{\mathbf{T}}^{\rm p} : \bar{\boldsymbol{\sigma}} \tag{69}$$

in which $\dot{\lambda}$ denotes the plastic consistency parameter.

Similar to Ju and Lee (2000), the effective equivalent plastic strain rate for the composite is defined as

$$\dot{\bar{e}}^{\rm p} \equiv \sqrt{\frac{2}{3}\dot{\bar{\boldsymbol{\epsilon}}}^{\rm p} : \bar{\mathbf{T}}^{p(-1)} : \dot{\bar{\boldsymbol{\epsilon}}}^{\rm p}} = 2(1-\phi_1)^2 \dot{\lambda}\sqrt{\frac{2}{3}\bar{\boldsymbol{\sigma}} : \bar{\mathbf{T}}^{\rm p} : \bar{\boldsymbol{\sigma}}} \tag{70}$$

The ensemble-volume averaged yield function in Eq. (68), the averaged plastic flow rule in Eq. (69), the equivalent plastic strain rate in Eq. (70), and the Kuhn–Tucker conditions completely characterize the effective plasticity formulation for a composite material with any isotropic hardening function $K(\bar{e}^{\rm p})$. It is feasible to extend the proposed model to accommodate kinematic hardening. In the following, the simple power-law type isotropic hardening function is employed as an example:

$$K(\bar{e}^{\rm p}) = \sqrt{\frac{2}{3}}\left\{\sigma_{\rm y} + h(\bar{e}^{\rm p})^{\bar{q}}\right\} \tag{71}$$

where $\sigma_{\rm y}$ is the initial yield stress, and h and $\bar{q}$ define the linear and exponential isotropic hardening parameters, respectively, for the composite.

4. Evolutionary probabilistic interfacial debonding

The progressive, partial interfacial debonding may occur under increasing deformations and affect the overall stress–strain behavior of PRDMCs. After the interfacial debonding between particles and the matrix, the debonded particles lose the load-carrying capacity along the debonded direction only and are regarded as partially debonded particles. Within the framework of the first-order (noninteracting) approximation, the stresses inside particles should be uniform. Following Tohgo and Weng (1994) and Zhao and Weng (1995, 1996, 1997), we employ the average internal stress of a particle as the controlling factor. The probability of partial particle debonding is modeled as a two-parameter Weibull process; see, e.g., Tohgo and Weng (1994), and Zhao and Weng (1995). Assuming that the Weibull (1951) statistics applies, we can express the cumulative probability distribution function of particle debonding, $P_{\rm d}$, for the uniaxial tensile loading (in the 1-direction) as

$$P_{\rm d}[(\bar{\sigma}_{11})_1] = 1 - \exp\left[-\left(\frac{(\bar{\sigma}_{11})_1}{S_0}\right)^M\right] \tag{72}$$

in which $(\bar{\sigma}_{11})_1$ is the internal stress of particles (phase 1) in the 1-direction, the subscript $(\cdot)_1$ signifies the particle phase, and S_0 and M are the well-known Weibull parameters.

Therefore, the current volume fraction of partially debonded particles ϕ_2 at a given level of $(\bar{\sigma}_{11})_1$ is given by

$$\phi_2 = \phi P_{\mathrm{d}}[(\bar{\sigma}_{11})_1] = \phi\left\{1 - \exp\left[-\left(\frac{(\bar{\sigma}_{11})_1}{S_0}\right)^M\right]\right\} \tag{73}$$

where ϕ is the original particle volume fraction.

The formulation of the internal stresses of particles needed to initiate particle debonding was previously investigated by Ju and Lee (2000); see Eqs. (61)–(68) therein. Here, as the simplest way among other possibilities to extend the above to triaxial loading, we could employ the major principal tensile loading direction (following a simple transformation at each loading step) as the 1-direction under triaxial loading histories. However, in a more general, variably multi-axial loading history in which the major principal tensile direction could change from step to step, we would have to perform new orthotropic or anisotropic formulation in Section 2 (instead of transverse isotropy).

5. Elastoplastic stress–strain relationship for partially debonded three-phase PRDMCs

In order to illustrate the proposed micromechanics-based elastoplastic damage model for PRDMCs, let us consider the example of uniaxial tensile loading here.

The applied macroscopic stress $\bar{\boldsymbol{\sigma}}$ can be written as

$$\bar{\sigma}_{11} \neq 0, \quad \text{all other } \bar{\sigma}_{ij} = 0 \tag{74}$$

With the simple isotropic hardening law described by Eq. (71), the overall yield function reads

$$\bar{F}(\bar{\boldsymbol{\sigma}}, \bar{e}^{\mathrm{p}}) = (1-\phi_1)^2 \bar{\boldsymbol{\sigma}} : \bar{\mathbf{T}}^{\mathrm{p}} : \bar{\boldsymbol{\sigma}} - \tfrac{2}{3}\left\{\sigma_{\mathrm{y}} + h(\bar{e}^{\mathrm{p}})^{\bar{q}}\right\}^2 \tag{75}$$

Substituting Eq. (74) into Eq. (75), the effective yield function of partially debonded three-phase PRDMCs under the uniaxial loading is obtained as

$$\bar{F} = (1-\phi_1)^2(\bar{T}_1^{\mathrm{p}} + 4\bar{T}_2^{\mathrm{p}} + \bar{T}_3^{\mathrm{p}} + \bar{T}_4^{\mathrm{p}} + \bar{T}_5^{\mathrm{p}} + 2\bar{T}_6^{\mathrm{p}})\bar{\sigma}_{11}^2 - \frac{2}{3}\left\{\sigma_{\mathrm{y}} + h(\bar{e}^{\mathrm{p}})^{\bar{q}}\right\}^2 \tag{76}$$

The macroscopic incremental plastic strain rate defined by Eq. (69) becomes

$$\Delta\bar{\boldsymbol{\epsilon}}^{\mathrm{p}} = 2(1-\phi_1)^2 \Delta\lambda \bar{\sigma}_{11} \begin{pmatrix} \bar{T}_1^{\mathrm{p}} + 4\bar{T}_2^{\mathrm{p}} + \bar{T}_3^{\mathrm{p}} + \bar{T}_4^{\mathrm{p}} + \bar{T}_5^{\mathrm{p}} + 2\bar{T}_6^{\mathrm{p}} & 0 & 0 \\ 0 & \bar{T}_3^{\mathrm{p}} + \bar{T}_5^{\mathrm{p}} & 0 \\ 0 & 0 & \bar{T}_3^{\mathrm{p}} + \bar{T}_5^{\mathrm{p}} \end{pmatrix} \tag{77}$$

for any stress beyond the initial yielding. Similarly, the incremental equivalent plastic strain can be written as

$$\Delta\bar{e}^{\mathrm{p}} = 2(1-\phi_1)^2 \Delta\lambda |\bar{\sigma}_{11}| \sqrt{\frac{2}{3}(\bar{T}_1^{\mathrm{p}} + 4\bar{T}_2^{\mathrm{p}} + \bar{T}_3^{\mathrm{p}} + \bar{T}_4^{\mathrm{p}} + \bar{T}_5^{\mathrm{p}} + 2\bar{T}_6^{\mathrm{p}})} \tag{78}$$

From Eq. (5), the macroscopic incremental elastic strain takes the form

$$\Delta\bar{\epsilon}^{\mathrm{e}} = \begin{bmatrix} \frac{k_*}{-l_*^2+k_*n_*} & 0 & 0 \\ 0 & \frac{l_*}{2(l_*^2-k_*n_*)} & 0 \\ 0 & 0 & \frac{l_*}{2(l_*^2-k_*n_*)} \end{bmatrix} \Delta\bar{\sigma}_{11} \tag{79}$$

Furthermore, the total incremental strain is the sum of the elastic incremental strain and plastic incremental strain.

By enforcing the plastic consistency condition $\bar{F} = 0$, the nonlinear equation is obtained for the uniaxial loading case as (cf. Eq. (78)):

$$\begin{aligned}&(1-\phi_1)^2(\bar{T}_1^{\mathrm{p}} + 4\bar{T}_2^{\mathrm{p}} + \bar{T}_3^{\mathrm{p}} + \bar{T}_4^{\mathrm{p}} + \bar{T}_5^{\mathrm{p}} + 2\bar{T}_6^{\mathrm{p}})\bar{\sigma}_{11}^2 \\ &= \frac{2}{3}\left\{\sigma_{\mathrm{y}} + h\sum_i \left[2(1-\phi_1)^2\Delta\lambda\sqrt{\frac{2}{3}(\bar{T}_1^{\mathrm{p}} + 4\bar{T}_2^{\mathrm{p}} + \bar{T}_3^{\mathrm{p}} + \bar{T}_4^{\mathrm{p}} + \bar{T}_5^{\mathrm{p}} + 2\bar{T}_6^{\mathrm{p}})}|\bar{\sigma}_{11}|\right]^{(i)\bar{q}}\right\}^2\end{aligned} \tag{80}$$

where the superscript i denotes the ith time-step value of a parameter.

In the case of a monotonic uniaxial tensile loading, the overall uniaxial stress–strain relation can be obtained by integrating Eqs. (77) and (79) as follows:

$$\begin{aligned}\bar{\epsilon} = &\sum_i \begin{bmatrix} \frac{k_*}{-l_*^2+k_*n_*} & 0 & 0 \\ 0 & \frac{l_*}{2(l_*^2-k_*n_*)} & 0 \\ 0 & 0 & \frac{l_*}{2(l_*^2-k_*n_*)} \end{bmatrix}^i \Delta\bar{\sigma}_{11}^i \\ &+ 2\sum_i [(1-\phi_1)^2\Delta\lambda\ \bar{\sigma}_{11}]^i \begin{pmatrix} \bar{T}_1^{\mathrm{p}} + 4\bar{T}_2^{\mathrm{p}} + \bar{T}_3^{\mathrm{p}} + \bar{T}_4^{\mathrm{p}} + \bar{T}_5^{\mathrm{p}} + 2\bar{T}_6^{\mathrm{p}} & 0 & 0 \\ 0 & \bar{T}_3^{\mathrm{p}} + \bar{T}_5^{\mathrm{p}} & 0 \\ 0 & 0 & \bar{T}_3^{\mathrm{p}} + \bar{T}_5^{\mathrm{p}} \end{pmatrix}^i\end{aligned} \tag{81}$$

The computational integration algorithms employed in this paper are very similar to those in Ju and Lee (2000); see Tables 1 and 2 therein. Therefore, we will not repeat the details here.

6. Numerical simulations and experimental comparison

In order to illustrate the influence of partially debonded particles on the behavior of ductile matrix composites, the present micromechanics-based predictions with varying S_0 values are compared with the predictions of Ju and Lee's (2000) complete particle debonding model (cf. Figs. 3 and 4) and Zhao and Weng's (1996) damage model (cf. Fig. 5). Specifically, Fig. 4 exhibits the evolution of volume fractions of partially debonded particles versus the uniaxial strains, which is corresponding to Fig. 3. For convenience, we adopt the same material parameters for the 6061-T6 aluminum alloy matrix/silicon-carbide particle composites as those used in Zhao and Weng (1996); see also Arsenault (1984), and Nieh and Chellman (1984). Therefore, we have $E_0 = 68.3$ GPa, $\nu_0 = 0.33$, $E_1 = 490$ GPa, $\nu_1 = 0.17$, $\sigma_{\mathrm{y}} = 250$ MPa, $h = 173$ MPa, $\bar{q} = 0.55$, $M = 5$, and $\phi = 0.2$. From Figs. 3 and 4, it is clear that the influence of partially debonded particles on the overall elastoplastic-damage behavior is more pronounced if S_0 is low. Further, higher interfacial strength parameter S_0 leads to higher stress–strain response. From Figs. 3 and 5, it is observed that the present predictions differ from those in Zhao and Weng (1996) due to the nature of two different formulations.

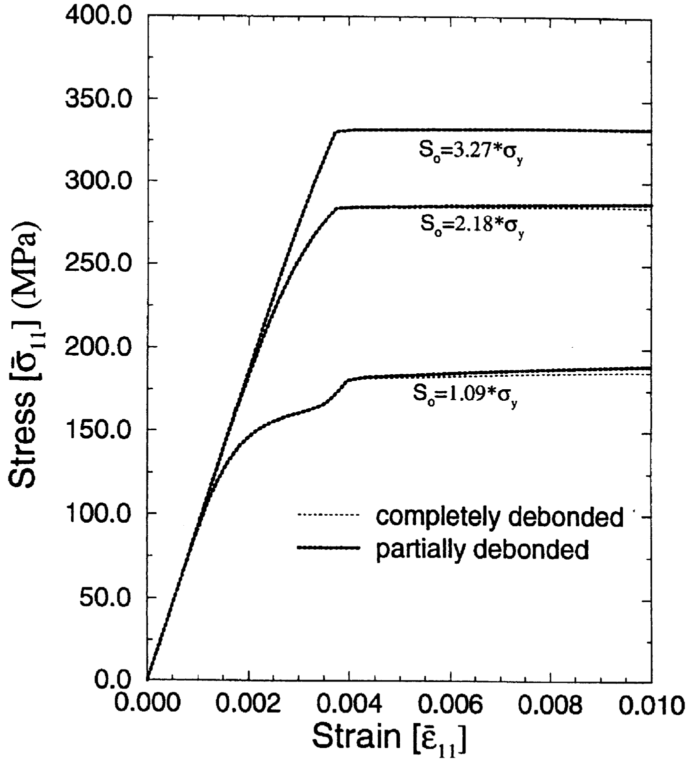

Fig. 3. The comparisons of predicted elastoplastic responses of PRDMCs between the present partially debonded damage model and Ju and Lee's (2000) completely debonded damage model with evolutionary debonding mechanism under uniaxial tension with $\phi = 0.2$, $M = 5$ and various S_0 values.

To assess the potential of the present framework further, we compare the present predictions with the experimental data reported by Llorca et al. (1991) for the uniaxial stress–strain behavior of Al–Cu matrix with SiC particulate composites. Here, we employ the same elastic properties according to Llorca et al. (1991) as follows: $E_0 = 71.8$ GPa, $\nu_0 = 0.33$, $E_1 = 450$ GPa and $\nu_1 = 0.17$. Both $\phi = 6\%$ and 13% are considered. Since the proposed effective plasticity model is different from that used in Llorca et al. (1991), we need to estimate the plastic parameters σ_y, h and $\bar{q}$ given in Eq. (71). Moreover, to implement the Weibull evolutionary debonding model, we need to estimate the two Weibull parameters S_0 and M. Here, we follow the parameter estimation algorithm developed by Ju et al. (1987) and Simo et al. (1988) to determine the proper values of σ_y, h, $\bar{q}$, S_0 and M. Based on the experimental data documented in Llorca et al. (1991), we estimate the above plastic and Weibull parameters to be: $\sigma_y = 169$ MPa, $h = 463.24$ MPa, $\bar{q} = 0.39252$, $S_0 = 3868.41$ MPa, and $M = 5$; cf. Ju and Lee (2000).

On the basis of the above material parameters, the present theory considering the partial debonding mechanism is exercised against the experimental data provided by Llorca et al. (1991) for the two uniaxial tests displayed in Fig. 6. Specifically, we consider various uniaxial stress–strain responses with and without the evolutionary debonding mechanism in Fig. 6. It is observed that the responses with the partial interfacial debonding model are lower than those without the damage mechanism. Furthermore, higher particle volume fraction leads to more significant differences between the partial debonding and perfect bonding models.

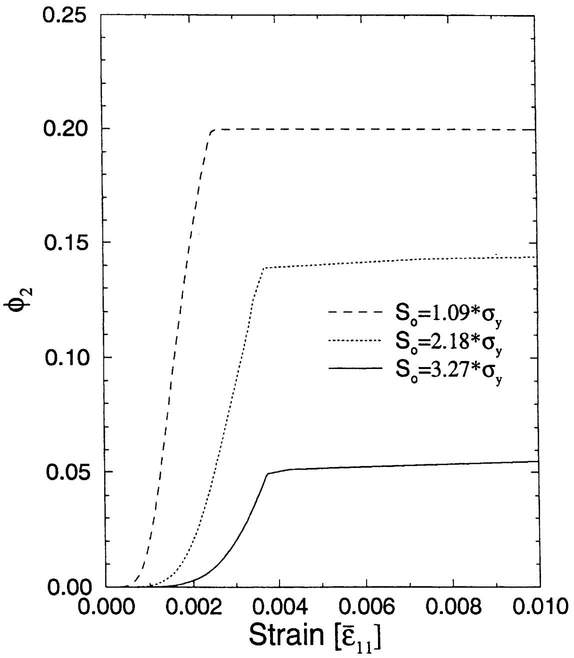

Fig. 4. The predicted evolution of volume fractions of partially debonded particles corresponding to Fig. 3.

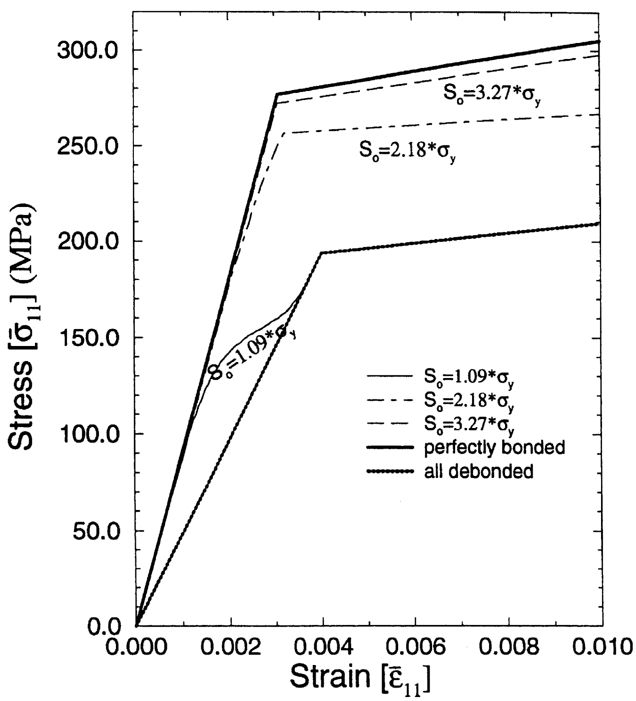

Fig. 5. Zhao and Weng's (1996) predicted elastoplastic responses of PRDMCs with evolutionary debonding damage under uniaxial tension with $\phi = 0.2$, $M = 5$ and various S_0 values.

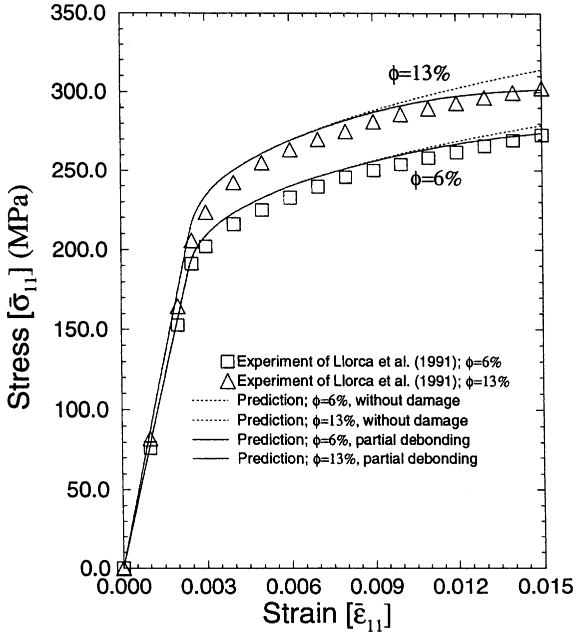

Fig. 6. The comparison of the present predictions with experimental data (Llorca et al., 1991) for overall uniaxial tensile responses of PRDMCs at initial particle volume fractions of 0.06 and 0.13.

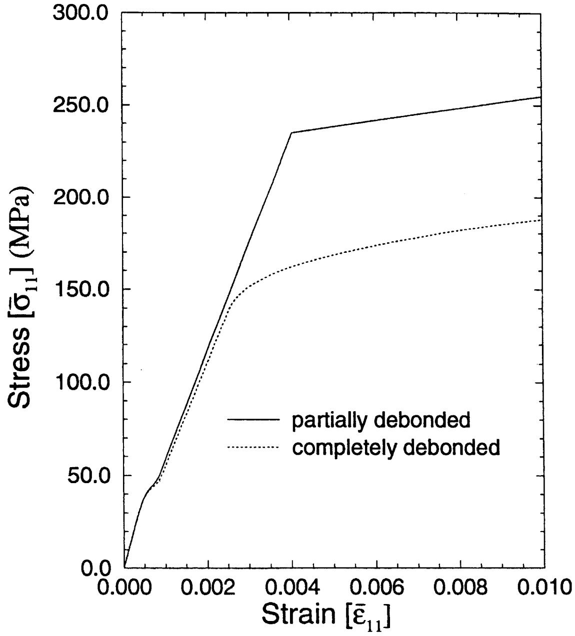

Fig. 7. The comparison of predicted evolutionary overall elastoplastic responses of PRDMCs between the present partial debonding model and Ju and Lee's (2000) complete debonding model with weaker interfacial strength $S_0 = 100$ MPa and $\phi = 0.2$.

Finally, to further investigate the influence of partially debonded particles on the overall elastoplastic-damage stress–strain responses, the value of the Weibull parameter S_0 (which governs the interfacial bonding strength) is decreased from 3868.41 to 100 MPa. Fig. 7 renders the numerical comparison between the proposed partial debonding model and Ju and Lee's (2000) complete debonding model with $\phi = 20\%$. Clearly, in comparison with the present formulation, the complete particle debonding model of Ju and Lee (2000) gives rise to substantially weaker overall stress–stain response when the interfacial bonding strength is lower and the particle volume fraction is higher.

7. Conclusion

A micromechanical elastoplastic-damage model considering progressive partial particle debonding is proposed to predict the overall stress–strain response and damage evolution of three-phase PRDMCs. To meet the characteristics of partially debonded interfaces, a partially debonded particle is replaced by an "equivalent", perfectly bonded transversely isotropic particle. The effective elastic moduli of the composite are then micromechanically derived. To estimate the overall elastoplastic behavior, an effective damage–yield criterion has been assumed and written in terms of quantities which are derived via micromechanics, based on the ensemble-volume averaging procedure and the first-order effects on eigenstrains due to elastic spherical inclusions. The resulting effective elastoplastic-damage–yield criterion, together with the assumed overall associative plastic flow rule and the hardening law, provides the basic foundation for the estimation of effective elastoplastic-damage behavior of PRDMCs. It is emphasized that the effects of random dispersion of inclusions and evolutionary partial particle debonding are considered in our framework.

Moreover, the effective elastoplastic behavior of partially debonded PRDMCs is investigated to assess the influence of partially debonded particles on the overall constitutive behavior. A damage mechanism based on a Weibull's statistical function is proposed to characterize the probability of partial particle debonding. The proposed elastoplastic-damage model is applied to the special case of uniaxial tensile loading to predict the corresponding stress–strain responses. The present results are compared with Ju and Lee's (2000) predictions considering complete particle debonding, Zhao and Weng's (1996) theory, and the experimental data reported by Llorca et al. (1991).

From Fig. 6, it is observed that the predicted overall stress–strain behavior of PRDMCs featuring partial particle debonding is in good qualitative agreement with the experimental data. In addition, when the interfacial particle bonding strength S_0 is weak and/or the initial particle volume fraction ϕ is medium, the influence of partially debonded particles on the overall stress–strain responses are rather significant. This is due to the relatively rapid damage evolution corresponding to weaker interfacial strength and/or higher volume fraction of debonded particles. By contrast, if the interfacial strength is high and the particle volume fraction is low, then the effects of partial particle debonding are not pronounced when compared with the complete particle debonding model or even the perfect bonding model.

Acknowledgements

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Appendix A. Parameters $c_1,\ldots, c_6$ in Eq. (26)

The parameters in Eq. (26) take the form:

$$c_1 = 2e_1\mu_0, \quad c_2 = 2e_2\mu_0, \quad c_3 = (e_1 + 4e_2 + 3e_3)(\kappa_0 - \tfrac{2}{3}\mu_0) + 2e_3\mu_0, \quad c_4 = 2e_4\mu_0,$$
$$c_5 = e_4\kappa_0 + 3\epsilon_5\kappa_0 + 2e_6\kappa_0 - \tfrac{2}{3}e_4\mu_0 - \tfrac{4}{3}e_6\mu_0, \quad c_6 = 2e_6\mu_0 \tag{A.1}$$

Here, we have

$$e_1 = 30(1 - v_0)\left[(j_1 + 4j_2 + d_3 + 2j_6)f_1\phi_2 + 4(j_1 + 2j_2 + j_3)f_2\phi_2 + (j_1 + 4j_2 + 3j_3)f_4\phi_2 + 2j_1\left(\frac{\phi_1}{4\beta} + f_6\phi_2\right)\right] \tag{A.2}$$

$$e_2 = 30(1 - v_0)\left[\frac{j_2\phi_1}{2\beta} + 2j_2f_2\phi_2 + 2j_6f_2\phi_2 + 2j_2f_6\phi_2\right] \tag{A.3}$$

$$e_3 = 30(1 - v_0)\left[(j_1 + 4j_2 + j_3 + 2j_6)f_3\phi_2 + (j_1 + 4j_2 + 3j_3)\left(-\frac{\alpha\phi_1}{6\alpha\beta + 4\beta^2} + f_5\phi_2\right) + 2j_3\left(\frac{\phi_1}{4\beta} + f_6\phi_2\right)\right] \tag{A.4}$$

$$e_4 = 30(1 - v_0)\left[(j_4 + j_5)f_1\phi_2 + 4(j_4 + j_5)f_2\phi_2 + (j_4 + 3j_5 + 2j_6)f_4\phi_2 + 2j_4\left(\frac{\phi_1}{4\beta} + f_6\phi_2\right)\right] \tag{A.5}$$

$$e_5 = 30(1 - v_0)\left[(j_4 + j_5)f_3\phi_2 + (j_4 + 3j_5 + 2j_6)\left(-\frac{\alpha\phi_1}{6\alpha\beta + 4\beta^2} + f_5\phi_2\right) + 2j_5\left(\frac{\phi_1}{4\beta} + f_6\phi_2\right)\right] \tag{A.6}$$

$$e_6 = \frac{1}{2} + 60(1 - v_0)\left[j_6\left(\frac{\phi_1}{4\beta} + f_6\phi_2\right)\right] \tag{A.7}$$

where the parameters $f_1,\ldots,f_6$ are the parameters of the fourth-rank tensor $\tilde{F}_{ijkl}(f_1, f_2, f_3, f_4, f_5, f_6)$ which is the inverse tensor of $\tilde{F}_{ijkl}(h_1, h_2, h_3, h_4, h_5, h_6)$ given in Eqs. (20)–(25). Furthermore, $j_1,\ldots,j_6$ are the parameters of the fourth-rank tensor $\tilde{F}_{ijkl}(j_1, j_2, j_3, j_4, j_5, j_6)$ which is the inverse tensor of $\tilde{F}_{ijkl}(g_1, g_2, g_3, g_4, g_5, g_6)$ with the following parameters:

$$g_1 = -4f_1(4 - 5v_0)\phi_2 \tag{A.8}$$

$$g_2 = -4f_2(4 - 5v_0)\phi_2 \tag{A.9}$$

$$g_3 = -2(-f_1 - 4f_2 + 5f_3 + 5f_1v_0 + 20f_2v_0 + 5f_3v_0)\phi_2 \tag{A.10}$$

$$g_4 = -4f_4(4 - 5v_0)\phi_2 \tag{A.11}$$

$$g_5 = -2\left[-\frac{\alpha(4 - 5v_0)}{3\alpha\beta + 2\beta^2}\phi_1 + (5v_0 - 1)\left(\frac{\phi_1}{2\beta} - \frac{3\alpha}{6\alpha\beta + 4\beta^2}\phi_1 + f_4\phi_2 + 2f_6\phi_2\right)\right] \tag{A.12}$$

$$g_6 = \frac{1}{2} - 4(4 - 5v_0)\left(\frac{\phi_1}{4\beta} + f_6\phi_2\right) \tag{A.13}$$

Appendix B. Parameters $w_1,\ldots, w_{17}$ in Eq. (38)

The parameters in Eq. (38) are explicitly expressed as

$$w_1 = 3u_5v_4 + 2u_6v_4 + u_4v_4 + 4u_5v_2 + u_5v_1 + u_4v_1\hat{n}_1^2 \tag{B.1}$$

$$w_2 = 2u_6v_3 \tag{B.2}$$

$$w_3 = 3u_3v_4 + 4u_2v_4 + u_1v_4 + 4u_3v_2 + u_3v_1 + u_1v_1\hat{n}_1^2 \tag{B.3}$$

$$w_4 = 2u_6v_1 \tag{B.4}$$

$$w_5 = 2u_3v_6 + 3u_3v_5 + 2u_2v_5 + u_1v_5 + 2u_2v_5 + u_3v_3 + u_1v_3\hat{n}_1^2 \tag{B.5}$$

$$w_6 = 2u_4v_6 \tag{B.6}$$

$$w_7 = 2u_5v_6 + 3u_5v_5 + 2u_6v_5 + u_4v_5 + u_5v_3 + u_4v_3\hat{n}_1^2 \tag{B.7}$$

$$w_8 = 2u_1v_6 \tag{B.8}$$

$$w_9 = 2u_6v_6 \tag{B.9}$$

$$w_{10} = 2u_2v_6 \tag{B.10}$$

$$w_{11} = 2u_2v_3\hat{n}_1 \tag{B.11}$$

$$w_{12} = 2u_2v_2 \tag{B.12}$$

$$w_{13} = 2u_6v_2 \tag{B.13}$$

$$w_{14} = 2u_2v_2\hat{n}_1 \tag{B.14}$$

$$w_{15} = 2u_4v_2\hat{n}_1 \tag{B.15}$$

$$w_{16} = 2u_1v_2\hat{n}_1 \tag{B.16}$$

$$w_{17} = 2u_2v_1\hat{n}_1 \tag{B.17}$$

Appendix C. The inner product of the fourth-rank tensor $\mathbf{F}^*$

The inner product between two fourth-rank tensors $\mathbf{F}^*(A_m)$ and $\mathbf{F}^*(B_m)$, $m = 1$–17, can be shown to follow the rule:

$$F^{\bullet}_{abcd}(Q_n) = F^*_{abpq}(A_m) \cdot F^*_{pqcd}(B_m) \tag{C.1}$$

where $\mathbf{F}^*$ is defined in Eq. (38). The components of the fourth-rank tensor $\mathbf{F}^{\bullet}$ take the form ($n = 1$–21)

$$F^{\bullet}_{abcd} = Q_1\tilde{n}_a\tilde{n}_b\delta_{cd} + Q_2\tilde{n}_c\tilde{n}_d\delta_{ab} + Q_3\tilde{n}_a\tilde{n}_b\tilde{n}_c\tilde{n}_d + Q_4(\tilde{n}_a\tilde{n}_b\hat{n}_c\hat{n}_d) + Q_5\tilde{n}_c\tilde{n}_d\hat{n}_a\hat{n}_b + Q_6(\tilde{n}_a\tilde{n}_b\tilde{n}_d\hat{n}_c + \tilde{n}_a\tilde{n}_b\tilde{n}_c\hat{n}_d) + Q_7(\tilde{n}_c\tilde{n}_d\tilde{n}_b\hat{n}_a + \tilde{n}_c\tilde{n}_d\tilde{n}_a\hat{n}_b) + Q_8(\tilde{n}_a\tilde{n}_c\delta_{bd} + \tilde{n}_a\tilde{n}_d\delta_{bc} + \tilde{n}_b\tilde{n}_c\delta_{ad} + \tilde{n}_b\tilde{n}_d\delta_{ac}) + Q_9(\tilde{n}_a\tilde{n}_c\hat{n}_b\hat{n}_d + \tilde{n}_a\tilde{n}_d\hat{n}_b\hat{n}_c + \tilde{n}_b\tilde{n}_c\hat{n}_a\hat{n}_d + \tilde{n}_b\tilde{n}_d\hat{n}_a\hat{n}_c) + Q_{10}\hat{n}_a\hat{n}_b\delta_{cd} + Q_{11}\hat{n}_c\hat{n}_d\delta_{ab} + Q_{12}(\hat{n}_a\hat{n}_c\delta_{bd} + \hat{n}_a\hat{n}_d\delta_{bc} + \hat{n}_b\hat{n}_c\delta_{ad} + \hat{n}_b\hat{n}_d\delta_{ac}) + Q_{13}\hat{n}_a\hat{n}_b\hat{n}_c\hat{n}_d + Q_{14}(\hat{n}_a\hat{n}_b\hat{n}_c\tilde{n}_d + \hat{n}_a\hat{n}_b\hat{n}_d\tilde{n}_c) + Q_{15}(\hat{n}_c\hat{n}_d\hat{n}_a\tilde{n}_b + \hat{n}_c\hat{n}_d\hat{n}_b\tilde{n}_a) + Q_{16}(\hat{n}_a\tilde{n}_b\delta_{cd} + \hat{n}_b\tilde{n}_a\delta_{cd}) + Q_{17}(\hat{n}_c\tilde{n}_d\delta_{ab} + \hat{n}_d\tilde{n}_c\delta_{ab}) + Q_{18}(\hat{n}_a\tilde{n}_c\delta_{bd} + \hat{n}_a\tilde{n}_d\delta_{bc} + \hat{n}_b\tilde{n}_c\delta_{ad} + \hat{n}_b\tilde{n}_d\delta_{ac}) + Q_{19}(\hat{n}_c\tilde{n}_a\delta_{bd} + \hat{n}_d\tilde{n}_a\delta_{bc} + \hat{n}_c\tilde{n}_b\delta_{ad} + \hat{n}_d\tilde{n}_b\delta_{ac}) + Q_{20}\delta_{ab}\delta_{cd} + Q_{21}(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \tag{C.2}$$

Here, the components $Q_1, \ldots, Q_{21}$ read

$$Q_1 = A_3B_1 + A_5B_1 + 4A_{14}B_{15} + 2A_{16}B_{15} + 4A_{10}B_6 + A_5B_6 + A_8B_6 + 2A_9B_6 + 4A_{10}B_7 + A_3B_7 + 3A_5B_7 + A_8B_7 + 2A_5B_9 + 2A_{16}B_1\hat{n}_1 + 4A_{10}B_{15}\hat{n}_1 + 2A_3B_{15}\hat{n}_1 + 2A_5B_{15}\hat{n}_1 + 2A_8B_{15}\hat{n}_1 + 4A_{14}B_6\hat{n}_1 + 2A_{16}B_6\hat{n}_1 + 2A_{16}B_7\hat{n}_1 + A_8B_1\hat{n}_1^2 + 2A_{16}B_{15}\hat{n}_1^2 + A_3B_6\hat{n}_1^2 \tag{C.3}$$

$$Q_2 = 4A_6B_{10} + 4A_7B_{10} + 4A_{15}B_{14} + 2A_{15}B_{16} + A_1B_3 + A_7B_3 + A_1B_5 + A_6B_5 + 3A_7B_5 + 2A_9B_5 + A_6B_8 + A_7B_8 + 2A_6B_9 + 4A_{15}B_{10}\hat{n}_1 + 4A_6B_{14}\hat{n}_1 + 2A_1B_{16}\hat{n}_1 + 2A_6B_{16}\hat{n}_1 + 2A_7B_{16}\hat{n}_1 + 2A_{15}B_3\hat{n}_1 + 2A_{15}B_5\hat{n}_1 + 2A_{15}B_8\hat{n}_1 + 2A_{15}B_{16}\hat{n}_1^2 + A_6B_3\hat{n}_1^2 + A_1B_8\hat{n}_1^2 \tag{C.4}$$

$$Q_3 = 4A_5B_{10} + 4A_8B_{10} + 4A_{16}B_{14} + 4A_{14}B_{16} + 2A_{16}B_{16} + A_3B_3 + A_5B_3 + 4A_{10}B_5 + A_3B_5 + 3A_5B_5 + A_8B_5 + 4A_{10}B_8 + A_5B_8 + A_8B_8 + 2A_9B_8 + 2A_8B_9 + 4A_{16}B_{10}\hat{n}_1 + 4A_8B_{14}\hat{n}_1 + 4A_{10}B_{16}\hat{n}_1 + 2A_3B_{16}\hat{n}_1 + 2A_5B_{16}\hat{n}_1 + 2A_8B_{16}\hat{n}_1 + 2A_{16}B_3\hat{n}_1 + 2A_{16}B_5\hat{n}_1 + 4A_{14}B_8\hat{n}_1 + 2A_{16}B_8\hat{n}_1 + 2A_{16}B_{16}\hat{n}_1^2 + A_8B_3\hat{n}_1^2 + A_3B_8\hat{n}_1^2 + 8A_{10}B_{10} \tag{C.5}$$

$$Q_4 = 4A_3B_{13} + 4A_5B_{13} + 4A_{10}B_2 + A_3B_2 + 3A_5B_2 + A_8B_2 + A_3B_4 + A_5B_4 + 2A_3B_9 + 4A_{16}B_{13}\hat{n}_1 + 2A_{16}B_2\hat{n}_1 + 2A_{16}B_4\hat{n}_1 + A_8B_4\hat{n}_1^2 \tag{C.6}$$

$$Q_5 = 4A_2B_{10} + 4A_{13}B_3 + A_2B_3 + A_4B_3 + 2A_9B_3 + 4A_{13}B_5 + 3A_2B_5 + A_4B_5 + A_2B_8 + 4A_{13}B_{16}\hat{n}_1 + 2A_2B_{16}\hat{n}_1 + 2A_4B_{16}\hat{n}_1 + A_4B_8\hat{n}_1^2 \tag{C.7}$$

$$Q_6 = 2A_{16}B_{10} + 4A_{10}B_{11} + A_3B_{11} + 3A_5B_{11} + A_8B_{11} + A_{16}B_{12} + 2A_{16}B_{13} + 2A_3B_{14} + 2A_5B_{14} + A_3B_{17} + A_5B_{17} + 2A_{16}B_9 + 2A_3B_{10}\hat{n}_1 + 2A_{16}B_{11}\hat{n}_1 + 2A_3B_{12}\hat{n}_1 + 2A_5B_{12}\hat{n}_1 + 2A_8B_{12}\hat{n}_1 + 2A_8B_{13}\hat{n}_1 + 2A_{16}B_{14}\hat{n}_1 + 2A_{16}B_{17}\hat{n}_1 + 2A_{16}B_{12}\hat{n}_1^2 + A_8B_{17}\hat{n}_1^2 + 2(2A_{10}B_{14} + 2A_{12}B_{14} + 2A_{12}B_{10}\hat{n}_1) \tag{C.8}$$

$$Q_7 = 4A_{11}B_{10} + 2A_{10}B_{16} + 2A_{12}B_{16} + 2A_{13}B_{16} + 2A_9B_{16} + A_{11}B_3 + 2A_{14}B_3 + A_{17}B_3 + 3A_{11}B_5 + 2A_{14}B_5 + A_{17}B_5 + A_{11}B_8 + 2A_{11}B_{16}\hat{n}_1 + 2A_{14}B_{16}\hat{n}_1 + 2A_{17}B_{16}\hat{n}_1 + 2A_{10}B_3\hat{n}_1 + 2A_{12}B_3\hat{n}_1 + 2A_{12}B_5\hat{n}_1 + 2A_{12}B_8\hat{n}_1 + 2A_{13}B_8\hat{n}_1 + 2A_{12}B_{16}\hat{n}_1^2 + A_{17}B_8\hat{n}_1^2 + 4(A_{10}B_{14} + A_{12}B_{14} + A_{12}B_{10}\hat{n}_1) \tag{C.9}$$

$$Q_8 = 2A_{10}B_{10} + 2A_9B_{10} + 2A_{14}B_{14} + 2A_{10}B_9 + 2A_{14}B_{10}\hat{n}_1 + 2A_{10}B_{14}\hat{n}_1 \tag{C.10}$$

$$\begin{aligned}Q_9 = {} & 2A_{12}B_{10} + 2A_{13}B_{10} + 2A_{10}B_{12} + 2A_{12}B_{12} + 2A_{13}B_{12} + 2A_9B_{12} + 2A_{10}B_{13} + 2A_{12}B_{13} \\ & + 2A_{12}B_9 + 2A_{14}B_{12}\hat{n}_1 + 2A_{12}B_{14}\hat{n}_1 + 3A_{11}B_{11} + 2A_{14}B_{11} + A_{17}B_{11} + 2A_{11}B_{14} + 2A_{17}B_{14} \\ & + A_{11}B_{17} + 2A_{14}B_{17} + A_{17}B_{17} + 2A_{17}B_{10}\hat{n}_1 + 2A_{12}B_{11}\hat{n}_1 + 2A_{11}B_{12}\hat{n}_1 + 2A_{17}B_{12}\hat{n}_1 \\ & + 2A_{10}B_{17}\hat{n}_1 + 2A_{12}B_{17}\hat{n}_1 + 2A_{14}B_{14} + 2A_{12}B_{12}\hat{n}_1^2 \end{aligned} \tag{C.11}$$

$$\begin{aligned}Q_{10} = {} & 4A_{13}B_1 + A_2B_1 + A_4B_1 + 2A_9B_1 + A_2B_6 + 4A_{13}B_7 + 3A_2B_7 + A_4B_7 + 2A_2B_9 + 4A_{13}B_{15}\hat{n}_1 \\ & + 2A_2B_{15}\hat{n}_1 + 2A_4B_{15}\hat{n}_1 + A_4B_6\hat{n}_1^2 \end{aligned} \tag{C.12}$$

$$\begin{aligned}Q_{11} = {} & 4A_1B_{13} + 4A_7B_{13} + A_1B_2 + A_6B_2 + 3A_7B_2 + 2A_9B_2 + A_1B_4 + A_7B_4 + 2A_1B_9 + 4A_{15}B_{13}\hat{n}_1 \\ & + 2A_{15}B_2\hat{n}_1 + 2A_{15}B_4\hat{n}_1 + A_6B_4\hat{n}_1^2 \end{aligned} \tag{C.13}$$

$$Q_{12} = 2A_{13}B_{13} + 2A_9B_{13} + 2A_{13}B_9 \tag{C.14}$$

$$\begin{aligned}Q_{13} = {} & 4A_2B_{13} + 4A_4B_{13} + 4A_{13}B_2 + 3A_2B_2 + A_4B_2 + 4A_{13}B_4 + A_2B_4 + A_4B_4 + 2A_9B_4 + 2A_4B_9 \\ & + 8A_{13}B_{13} \end{aligned} \tag{C.15}$$

$$\begin{aligned}Q_{14} = {} & 4A_{13}B_{11} + 3A_2B_{11} + A_4B_{11} + 2A_2B_{14} + 2A_4B_{14} + 4A_{13}B_{17} + A_2B_{17} + A_4B_{17} + 2A_9B_{17} \\ & + 2A_4B_{10}\hat{n}_1 + 2A_2B_{12}\hat{n}_1 + 2A_4B_{12}\hat{n}_1 + 4(A_{13}B_{14} + A_{13}B_{12}\hat{n}_1) \end{aligned} \tag{C.16}$$

$$\begin{aligned}Q_{15} = {} & 4A_{11}B_{13} + 4A_{17}B_{13} + 3A_{11}B_2 + 2A_{14}B_2 + A_{17}B_2 + A_{11}B_4 + 2A_{14}B_4 + A_{17}B_4 + 2A_{17}B_9 \\ & + 2A_{12}B_2\hat{n}_1 + 2A_{10}B_4\hat{n}_1 + 2A_{12}B_4\hat{n}_1 + 4(A_{13}B_{14} + A_{13}B_{12}\hat{n}_1) \end{aligned} \tag{C.17}$$

$$\begin{aligned}Q_{16} = {} & A_{11}B_1 + 2A_{14}B_1 + A_{17}B_1 + 2A_{10}B_{15} + 2A_{12}B_{15} + 2A_{13}B_{15} + 2A_9B_{15} + A_{11}B_6 + 3A_{11}B_7 \\ & + 2A_{14}B_7 + A_{17}B_7 + 2A_{11}B_9 + 2A_{10}B_1\hat{n}_1 + 2A_{12}B_1\hat{n}_1 + 2A_{11}B_{15}\hat{n}_1 + 2A_{14}B_{15}\hat{n}_1 \\ & + 2A_{17}B_{15}\hat{n}_1 + 2A_{12}B_6\hat{n}_1 + 2A_{13}B_6\hat{n}_1 + 2A_{12}B_7\hat{n}_1 + 2A_{12}B_{15}\hat{n}_1^2 + A_{17}B_6\hat{n}_1^2 \end{aligned} \tag{C.18}$$

$$\begin{aligned}Q_{17} = {} & 2A_{15}B_{10} + A_1B_{11} + A_6B_{11} + 3A_7B_{11} + 2A_9B_{11} + 2A_{15}B_{12} + 2A_{15}B_{13} + 2A_1B_{14} + 2A_7B_{14} \\ & + A_1B_{17} + A_7B_{17} + 2A_{15}B_9 + 2A_1B_{10}\hat{n}_1 + 2A_{15}B_{11}\hat{n}_1 + 2A_1B_{12}\hat{n}_1 + 2A_6B_{12}\hat{n}_1 + 2A_7B_{12}\hat{n}_1 \\ & + 2A_6B_{13}\hat{n}_1 + 2A_{15}B_{14}\hat{n}_1 + 2A_{15}B_{17}\hat{n}_1 + 2A_{15}B_{12}\hat{n}_1^2 + A_6B_{17}\hat{n}_1^2 \end{aligned} \tag{C.19}$$

$$Q_{18} = 2A_{13}B_{14} + 2A_9B_{14} + 2A_{13}B_{10}\hat{n}_1 \tag{C.20}$$

$$Q_{19} = 2A_{14}B_{13} + 2A_{14}B_9 + 2A_{10}B_{13}\hat{n}_1 \tag{C.21}$$

$$\begin{aligned}Q_{20} = {} & A_1B_1 + A_7B_1 + 2A_{15}B_{15} + A_6B_6 + A_7B_6 + A_1B_7 + A_6B_7 + 3A_7B_7 + 2A_9B_7 + 2A_7B_9 \\ & + 2A_{15}B_1\hat{n}_1 + 2A_1B_{15}\hat{n}_1 + 2A_6B_{15}\hat{n}_1 + 2A_7B_{15}\hat{n}_1 + 2A_{15}B_6\hat{n}_1 + 2A_{15}B_7\hat{n}_1 + A_6B_1\hat{n}_1^2 \\ & + 2A_{15}B_{15}\hat{n}_1^2 + A_1B_6\hat{n}_1^2 \end{aligned} \tag{C.22}$$

$$Q_{21} = 2A_9B_9 \tag{C.23}$$

Appendix D. Parameters $T_1^p, \ldots, T_6^p$ in Eq. (55)

The parameters in Eq. (55) are given by

$$T_1^{\mathrm{p}} = \frac{\phi_2}{35}\left\{ \frac{1680}{2}(a_2 + 5a_4)\mu_0 + 2880\mu_0^2(7a_1^2 + 56a_1a_2 + 28a_2^2 + 22a_2a_4 + 42a_1a_6 + 26a_2a_6 + 30a_4a_6) \right.$$
$$+ 13440\mu_0^2(-3a_1^2 - 24a_1a_2 - 12a_2^2 + 24a_2a_4 - 18a_2a_5 - 18a_1a_6 - 26a_2a_6 - 10a_4a_6 + 6a_2a_4v_0$$
$$- 18a_2a_5v_0 - 8a_2a_6v_0 - 4a_4a_6v_0) + (25a_1^2 + 200a_1a_2 + 160a_2^2 + 30a_1a_3 + 120a_2a_3 + 45a_3^2$$
$$+ 200a_2a_4 + 240a_2a_5 + 120a_1a_6 + 220a_2a_6 - 10a_1^2v_0 - 80a_1a_2v_0 + 80a_2^2v_0 + 60a_1a_3v_0$$
$$+ 240a_2a_3v_0 + 90a_3^2v_0 - 80a_2a_4v_0 + 120a_2a_5v_0 - 120a_1a_6v_0 + 80a_2a_6v_0 + 120a_4a_6v_0$$
$$+ 19a_1^2v_0^2 + 152a_1a_2v_0^2 + 136a_2^2v_0^2 + 30a_1a_3v_0^2 + 120a_2a_3v_0^2 + 45a_3^2v_0^2 - 28a_2a_4v_0^2$$
$$\left. - 120a_2a_5v_0^2 + 84a_1a_6v_0^2 - 104a_2a_6v_0^2 - 60a_4a_6v_0^2)\frac{5600\mu_0^2}{3} \right\} \tag{D.1}$$

$$T_2^{\mathrm{p}} = \frac{\phi_2}{35}\left\{ \frac{2520}{2}a_2\mu_0 + 1440a_2(153a_2 + 194a_6)\mu_0^2 + 6720a_2\mu_0^2(-67a_2 - 6a_6 + 2a_2v_0 - 108a_6v_0) \right.$$
$$\left. + \frac{11200}{2}a_2\mu_0^2(80a_2 + 40a_6 - 90a_2v_0 - 100a_6v_0 + 61a_2v_0^2 + 178a_6v_0^2) \right\} \tag{D.2}$$

$$T_3^{\mathrm{p}} = \frac{\phi_2}{35}\left\{ \frac{-8400}{2}a_2\mu_0 + 2880\mu_0^2(7a_1a_4 + 17a_2a_4 - 7a_1a_6 - 13a_2a_6 + 6a_4a_6) + 13440\mu_0^2(-3a_1a_4 + 9a_2a_5 \right.$$
$$+ 3a_1a_6 - 7a_2a_6 - 4a_4a_6 - 3a_2a_4v_0 + 9a_2a_5v_0 + 32a_2a_6v_0 + 2a_4a_6v_0) + (25a_1a_4 + 15a_3a_4$$
$$+ 15a_1a_5 - 60a_2a_5 + 45a_3a_5 - 10a_1a_6 + 30a_2a_6 + 30a_3a_6 + 60a_4a_6 - 10a_1a_4v_0 + 30a_3a_4v_0$$
$$+ 30a_1a_5v_0 + 60a_2a_5v_0 + 90a_3a_5v_0 + 40a_1a_6v_0 + 60a_3a_6v_0 - 120a_4a_6v_0 + 19a_1a_4v_0^2$$
$$+ 90a_2a_4v_0^2 + 15a_3a_4v_0^2 + 15a_1a_5v_0^2 + 120a_2a_5v_0^2 + 45a_3a_5v_0^2 - 4a_1a_6v_0^2 - 48a_2a_6v_0^2 + 30a_3a_6v_0^2$$
$$\left. + 72a_4a_6v_0^2)\frac{5600\mu_0^2}{3} \right\} \tag{D.3}$$

$$T_4^{\mathrm{p}} = \frac{\phi_2}{35}\left\{ \frac{-8400}{2}a_4\mu_0 + 2880(7a_1a_4 + 17a_2a_4 - 7a_1a_6 - 13a_2a_6 + 6a_4a_6)\mu_0^2 + 13340\mu_0^2(-3a_1a_4 + 9a_2a_5 \right.$$
$$+ 3a_1a_6 - 7a_2a_6 - 4a_4a_6 - 3a_2a_4v_0 + 9a_2a_5v_0 + 32a_2a_6v_0 + 2a_4a_6v_0) + (25a_1a_4 + 15a_3a_4$$
$$+ 15a_1a_5 - 60a_2a_5 + 45a_3a_5 - 10a_1a_6 + 30a_2a_6 + 30a_3a_6 + 60a_4a_6 - 10a_1a_4v_0 + 30a_3a_4v_0$$
$$+ 30a_1a_5v_0 + 60a_2a_5v_0 + 90a_3a_5v_0 + 40a_1a_6v_0 + 60a_3a_6v_0 - 120a_4a_6v_0 + 19a_1a_4v_0^2$$
$$+ 90a_2a_4v_0^2 + 15a_3a_4v_0^2 + 15a_1a_5v_0^2 + 120a_2a_5v_0^2 + 45a_3a_5v_0^2 - 4a_1a_6v_0^2 - 48a_2a_6v_0^2$$
$$\left. + 30a_3a_6v_0^2 + 72a_4a_6v_0^2)\frac{5600\mu_0^2}{3} \right\} \tag{D.4}$$

$$T_5^{\mathrm{p}} = -\frac{1}{3} + \frac{\phi_2}{35}\left\{ \frac{40320}{2}(a_4^2 - 2a_4a_6 - 2a_6^2)\mu_0^2 + 40320\mu_0^2(-a_4^2 + 2a_4a_6 + 2a_6^2) + (25a_4^2 + 30a_4a_5 + 45a_5^2 - 20a_4a_6 + 60a_5a_6 - 20a_6^2 - 10a_4^2\nu_0 + 60a_4a_5\nu_0 + 90a_5^2\nu_0 + 80a_4a_6\nu_0 + 120a_5a_6\nu_0 + 80a_6^2\nu_0 + 19a_4^2\nu_0^2 + 30a_4a_5\nu_0^2 + 45a_5^2\nu_0^2 - 8a_4a_6\nu_0^2 + 60a_5a_6\nu_0^2 - 8a_6^2\nu_0^2)\frac{5600\mu_0^2}{3} \right\} + \frac{200}{3}(1-2\nu_0)^2\frac{\phi_1}{(3\alpha+2\beta)^2} - 2(23 - 50\nu_0 + 35\nu_0^2)\frac{\phi_1}{\beta^2} \tag{D.5}$$

$$T_6^{\mathrm{p}} = \frac{1}{2} + \frac{\phi_2}{35}\left\{ \frac{-4200}{2}a_6\mu_0 + 99360a_6^2\mu_0^2 + 6720a_6^2\mu_0^2(-31 + 2\nu_0) + 5600a_6^2\mu_0^2(40 - 50\nu_0 + \frac{66}{2}\nu_0^2) \right\} + (23 - 50\nu_0 + 35\nu_0^2)\frac{\phi_1}{\beta^2} \tag{D.6}$$

In the above equations, the parameters $a_1,\ldots,a_6$ are given by

$$a_1 = \frac{\mathscr{A}}{8h_6(h_2+h_6)(-h_3h_4 + h_1h_5 + 4h_2h_5 + h_1h_6 + 4h_2h_6 + h_3h_6 + h_4h_6 + 3h_5h_6 + 2h_6h_6)\mu_0} \tag{D.7}$$

$$a_2 = \frac{-h_2}{8h_6(h_2+h_6)\mu_0} \tag{D.8}$$

$$a_3 = \frac{-h_3h_4 + h_1h_5 + 4h_2h_5 - 2h_3h_6}{8h_6(-h_3h_4 + h_1h_5 + 4h_2h_5 + h_1h_6 + 4h_2h_6 + h_3h_6 + h_4h_6 + 3h_5h_6 + 2h_6h_6)\mu_0} \tag{D.9}$$

$$a_4 = \frac{\mathscr{B}}{72h_6(-h_3h_4 + h_1h_5 + 4h_2h_5 + h_1h_6 + 4h_2h_6 + h_3h_6 + h_4h_6 + 3h_5h_6 + 2h_6h_6)\kappa_0\mu_0} \tag{D.10}$$

$$a_5 = \frac{\mathscr{C}}{72h_6(-h_3h_4 + h_1h_5 + 4h_2h_5 + h_1h_6 + 4h_2h_6 + h_3h_6 + h_4h_6 + 3h_5h_6 + 2h_6h_6)\kappa_0\mu_0} \tag{D.11}$$

$$a_6 = \frac{1}{8h_6\mu_0} \tag{D.12}$$

and the coefficients $\mathscr{A}$,$\mathscr{B}$ and $\mathscr{C}$ read:

$$\mathscr{A} = -h_2h_3h_4 + h_1h_2h_5 + 4h_2h_2h_5 + 2h_1h_2h_6 + 8h_2h_2h_6 + 4h_2h_3h_6 + 4h_2h_4h_6 + 3h_3h_4h_6 - 3h_1h_5h_6 - 2h_1h_6h_6 \tag{D.13}$$

$$\mathscr{B} = -9h_3h_4\kappa_0 + 9h_1h_5\kappa_0 + 36h_2h_5\kappa_0 + 6h_1h_6\kappa_0 + 24h_2h_6\kappa_0 - 4h_1h_6\mu_0 - 16h_2h_6\mu_0 - 12h_4h_6\mu_0 \tag{D.14}$$

$$\mathscr{C} = +9h_3h_4\kappa_0 - 9h_1h_5\kappa_0 - 36h_2h_5\kappa_0 - 6h_1h_6\kappa_0 - 24h_2h_6\kappa_0 - 6h_4h_6\kappa_0 - 18h_5h_6\kappa_0 - 12h_6h_6\kappa_0 + 4h_1h_6\mu_0 + 16h_2h_6\mu_0 + 4h_4h_6\mu_0 + 8h_6h_6\mu_0 \tag{D.15}$$

Appendix E. Parameters $P_1^{\mathrm{p}},\ldots,P_6^{\mathrm{p}}$ in Eq. (57)

The parameters $P_1^{\mathrm{p}},\ldots,P_6^{\mathrm{p}}$ in Eq. (57) are the components of the fourth-rank tensor $\tilde{F}_{ijkl}(P_1^{\mathrm{p}},P_2^{\mathrm{p}},P_3^{\mathrm{p}},P_4^{\mathrm{p}},P_5^{\mathrm{p}},P_6^{\mathrm{p}})$, which is the inverse of $\tilde{F}_{ijkl}(d_1,d_2,d_3,d_4,d_5,d_6)$, with the following components

$$d_1 = \frac{\mathscr{D}(-7+5\nu_0)\phi_2}{2h_6(h_2+h_6)(-h_3h_4+h_1h_5+4h_2h_5+h_1h_6+4h_2h_6+h_3h_6+h_4h_6+3h_5h_6+2h_6^2)} \tag{E.1}$$

$$d_2 = \frac{h_2(7-5\nu_0)\phi_2}{2h_6(h_2+h_6)} \tag{E.2}$$

$$d_3 = \frac{\mathscr{E}\phi_2}{2h_6(-h_3h_4+h_1h_5+4h_2h_5+h_1h_6+4h_2h_6+h_3h_6+h_4h_6+3h_5h_6+2h_6^2)} \tag{E.3}$$

$$d_4 = \frac{(-h_3h_4+h_1h_5+4h_2h_5-2h_4h_6)(-7+5\nu_0)\phi_2}{2h_6(-h_3h_4+h_1h_5+4h_2h_5+h_1h_6+4h_2h_6+h_3h_6+h_4h_6+3h_5h_6+2h_6^2)} \tag{E.4}$$

$$d_5 = \frac{(-7\alpha+2\beta+5\alpha\nu_0-10\beta\nu_0)\phi_1}{3\alpha\beta+2\beta^2} + \frac{\mathscr{F}\phi_2}{2h_6(-h_3h_4+h_1h_5+4h_2h_5+h_1h_6+4h_2h_6+h_3h_6+h_4h_6+3h_5h_6+2h_6^2)} \tag{E.5}$$

$$d_6 = \frac{1}{2} + \frac{(7-5\nu_0)\phi_1}{2\beta} + \frac{(-7+5\nu_0)\phi_2}{2h_6} \tag{E.6}$$

Here, the coefficients $\mathscr{D}$, $\mathscr{E}$ and $\mathscr{F}$ read

$$\mathscr{D} = -h_2h_3h_4 + h_1h_2h_5 + 4h_2^2h_5 + 2h_1h_2h_6 + 8h_2^2h_6 + 4h_2h_3h_6 + 4h_2h_4h_6 + 3h_3h_4h_6 - 3h_1h_5h_6 + 2h_1h_6^2 \tag{E.7}$$

$$\mathscr{E} = 7h_3h_4 - 7h_1h_5 - 28h_2h_5 + 2h_1h_6 + 8h_2h_6 + 20h_3h_6 - 5h_3h_4\nu_0 + 5h_1h_5\nu_0 + 20h_2h_5\nu_0 - 10h_1h_6\nu_0 - 40h_2h_6\nu_0 - 40h_3h_6\nu_0 \tag{E.8}$$

$$\mathscr{F} = -7h_3h_4 + 7h_1h_5 + 28h_2h_5 - 2h_1h_6 - 8h_2h_6 - 2h_3h_6 + 14h_5h_6 - 4h_6^2 + 5h_3h_4\nu_0 - 5h_1h_5\nu_0 - 20h_2h_5\nu_0 + 10h_1h_6\nu_0 + 40h_2h_6\nu_0 + 10h_3h_6\nu_0 - 10h_5h_6\nu_0 + 20h_6^2\nu_0 \tag{E.9}$$

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